Topology of the Order Parameter in the Little-Parks Experiment

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Under appropriate geometry, magnetic field, and temperature, numerical minimization of the free energy predicts a stable superconducting phase where part of the sample is normal, so that the magnetic flux is not enclosed by the superconducting part. This phase mediates between the normal phase and the superconducting phase which has been usually considered. For one point in the filed-temperature plane, it has been proven analytically that this intermediate phase minimizes the free energy. Near the transition, even when the doubly connected phase is stable, the order parameter tries to mimic that of a simply connected phase.

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The Little-Parks experiment [1] deals with the onset of superconductivity in a cylindrical shell that encloses a magnetic flux. For a brief (not comprehensive) survey of the literature on this experiment see, e.g., Ref. [2].

The free energy of the sample can be expressed by means of the Ginzburg-Landau theory [3]. If we write the order parameter in the form $\psi = |\psi| e^{i\varphi}$, then the free-energy density of a superconducting phase, relative to the normal phase, is

$$-\alpha|\psi|^2 + \tfrac{1}{2}\beta|\psi|^4 + (\mathbf{H} - \mathbf{H}_{\mathrm{ext}})^2/8\pi + [\hbar^2(\nabla|\psi|)^2 + (\hbar\nabla\varphi - e^*\mathbf{A}/c)^2|\psi|^2]/2m^*, \tag{1}$$

where e^* is the charge of a Cooper pair, $\mathbf{H} = \nabla \times \mathbf{A}$ is the magnetic field, $\mathbf{H}_{\mathrm{ext}}$ is the magnetic field which would be present in the absence of superconductivity, and α, β, and m^* are positive parameters, which characterize the superconductive material. While the temperature dependence of β and m^* is not significant, $\alpha \to 0$ as $T \to T_c$ (critical temperature for bulk sample without magnetic field).

For the purpose of comparison with "classical" literature, we consider an essentially cylindrical shell of superconducting material with average radius R and *nonuniform* width $D(\theta)$, with θ the cylindrical angle. As usual, the axis of the cylinder is parallel to the magnetic field. $\xi = (\hbar^2/2m^*\alpha)^{1/2}$ is the coherence length and λ the penetration depth. For $\mathbf{H}_{\mathrm{ext}}$ uniform, $D(\theta) \ll \xi, \lambda$, and a cylinder either very long or very short relative to ξ, ψ becomes a function of θ only. The present analysis will be restricted to the case in which $D(\theta)$ and R are sufficiently small and λ sufficiently large to render the contribution of $\mathbf{H} - \mathbf{H}_{\mathrm{ext}}$ in Eq. (1) negligible.

Let us denote by g the free energy per unit length of the cylinder, relative to the normal phase. Using Eq. (1), requiring ψ to be single valued and the supercurrent $(e^*/m^*)|\psi|^2|\hbar\nabla\varphi - e^*\mathbf{A}/c|D(\theta)$ independent of θ, and writing $|\psi| = (\alpha/\beta)^{1/2}y(\theta)$, g can be brought to the expression

$$\beta g/\alpha^2 R = \int_0^{2\pi}[-y^2 + \tfrac{1}{2}y^4 + (\xi/R)^2(dy/d\theta)^2]D\,d\theta + (2\pi K\xi/R)^2 \Big/ \int_0^{2\pi} y^{-2}D^{-1}\,d\theta, \tag{2}$$

where K is the deviation from an integer number ($|K| \leq \frac{1}{2}$) of the magnetic flux inside the cylinder, measured in quantum units hc/e^*. Equation (2) shows that the material-dependent parameters merely provide the scaling for a universal behavior.

The last term on the right hand side (r.h.s.) of Eq. (2) is the thermodynamic price that a superconducting phase has to pay for being connected. This leads to the idea that g might be lowered by breaking the connection, namely, by requiring that $y(\theta)$ vanishes for some value of θ. On the other hand, introducing a constraint raises the value of the first term on the r.h.s. of Eq. (2), so that a detailed calculation is needed in order to know whether this constraint is thermodynamically favorable or not.

We shall call "singly connected phase" the situation in which the material is superconducting in a singly connected domain [$y(\theta) = 0$ at some θ, thus not enclosing the magnetic field] and, respectively, "doubly connected phase" the situation in which [$y(\theta) > 0$ for $0 \leq \theta \leq 2\pi$]. Calculations for $D(\theta) =$ const show that in this case the free energy g_s of the singly connected phase (SCP) is always higher than the free energy g_d of the doubly connected phase, even for the case $|K| = \frac{1}{2}$. Nevertheless, both g_s and g_d become zero simultaneously when $\xi \to 2R$. This result suggests that if at a given value of θ (call it $\theta = 0$) there is a "weak link," i.e., smaller α, ξ, or D, then the thermodynamic price of requiring $y(0) = 0$ should lessen and there should be a temperature range where the SCP is stable. In the present analysis we consider a uniform material and let D be a function of θ.

The Euler-Lagrange equation, which must be fulfilled when $y(\theta)$ minimizes (2), is

$$y'' + (D'/D)y' + (R/\xi)^2(y - y^3) - (2\pi K/D\Lambda)^2 y^{-3} = 0, \tag{3}$$

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with $\Lambda = \int_0^{2\pi} y^{-2} D^{-1}\, d\theta$.

Numerical study.—Since in general we are unable to solve the integro-differential equation (3), we minimized (2) for an explicit model. We consider

$$D(\theta) = D_0 + D_1 \sin(\theta/2)\,. \qquad (4)$$

We know from (3) that, sufficiently close to $\theta = 0$, $y(\theta)$ is of the form

$$y(\theta) = A_1[\theta + (2D_0/D_1)\exp(-\tfrac{1}{2}D_1\theta/D_0)] + A_2 \qquad (5a)$$

for the doubly connected phase, and

$$y(\theta) = A_1 \exp[-D_1\theta/(4D_0)] \times \sin\{(\theta R/\xi)[1 - (D_1\xi/4D_0R)^2]^{1/2}\} \qquad (5b)$$

for the SCP.

The technique for calculating $y(\theta)$ is as follows. Taking $y(2\pi - \theta) = y(\theta)$, we consider the range $0 \le \theta \le \pi$. For a certain θ^*, Eq. (5) is used in the range $0 \le \theta \le \theta^*$ and for $\theta > \theta^*$ we use the expansion

$$y(\theta) = [a_1 + a_2 \sin(\theta/2)] \times \left[1 + \sum_{j=3}^{n} a_j[1 - \sin(\theta/2)]^{j-2}\right]^{-1}, \qquad (6)$$

which should be able to approximate any well-behaved function for sufficiently large n. In the case of Eq. (5b), A_1 is determined by requiring continuity of y at $\theta = \theta^*$. In the case of Eq. (5a), continuity is also required for y'. The free energy is then minimized with respect to the parameters $a_1, \ldots, a_n$ and θ^*. This procedure was repeated for increasing values of n, until a further increase did not have a significant influence on the results.

The results described here correspond to $|K| = \frac{1}{2}$ and $D_1/D_0 = 0.15$. An imperfection of this order has most probably been present in many of the experiments reported in the literature, but has not been considered to modify essentially the results that would be obtained for $D_1 = 0$. Figure 1 shows $y(0)/y(\pi) = |\psi(0)/\psi(\pi)|$ as a function of ξ. In the "classical case" $D_1 = 0$, y is independent of θ, and $y(0)/y(\pi) = 1$ for every ξ. In our case, the order parameter has to lessen the second term on the r.h.s. of Eq. (2), and $y(0)/y(\pi)$ drops pronouncedly near $\xi \sim 1.85R$. In Fig. 2 the free energies of the singly and doubly connected phases are compared. At $\xi \sim 1.88R$, g_s becomes lower than g_d and a first order transition occurs. The arrow in Fig. 2 shows the value of ξ at which g_s becomes zero and a second order transition into the normal phase occurs.

According to these results, there exists a singly connected superconducting phase which mediates between the normal phase and the usually considered doubly connected phase. As in the case of the intermediate state or the case of vortices, some parts of the sample stay normal in order to achieve minimization of the overall free energy, but in the present case suppression of superconductivity in the appropriate places is induced by magnetic flux rather than by magnetic field. Denoting the normal-SCP transition temperature by T_s and the singly-doubly connected transition temperature by T_d, the results for the considered case indicate $T_s - T_d \approx 0.2(T_c - T_s)$. When D_1/D_0 is increased, the same qualitative behavior is found and the ratio $(T_s - T_d)/(T_c - T_s)$ increases. When $|K|$ deviates from $\frac{1}{2}$, $(T_s - T_d)/(T_c - T_s)$ decreases abruptly. (For $D_1 = D_0$, the SCP is not found for $\frac{1}{2} - |K| > 0.0017$.)

Analytic study.—The numerical study does not provide a picture for the influence of the shape of $D(\theta)$. Moreover, in the interesting region where $g_d \approx g_s$, a large

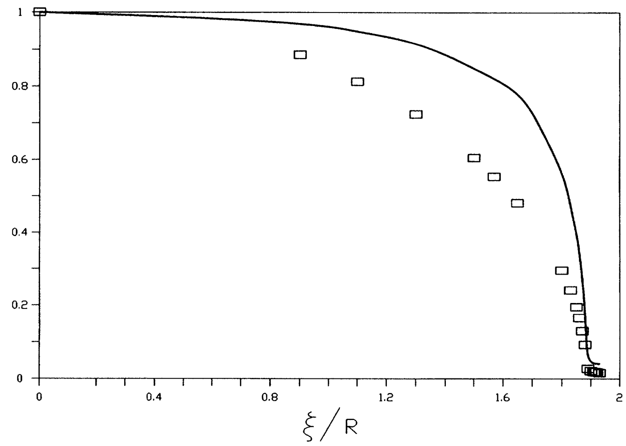

FIG. 1. Nonuniformity of the order parameter ψ as function of the coherence length. The full line describes the ratio $|\psi(0)/\psi(\pi)|$; the squares describe $(\beta/\alpha)^{1/2}|\psi(0)|$. The results shown here refer to the shell thickness given by Eq. (4), with $D_1/D_0 = 0.15$.

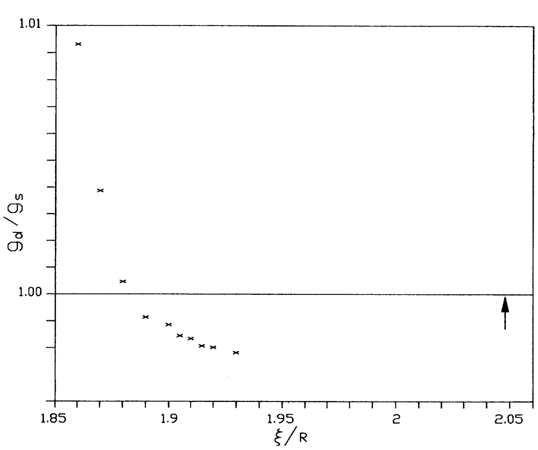

FIG. 2. Comparison between the free energies of the singly and doubly connected phases (relative to the normal phase). g_d (g_s) is more negative for $\xi < 1.88R$ ($> 1.88R$). As $\xi \to 2.05R$, $|\psi(\theta)| \to 0$.

number of parameters ($n \sim 20$) is needed to represent $y(\theta)$ in the doubly connected case. Therefore, there is some risk that g_d was found larger than g_s due to imperfect minimization.

Analytic treatment is possible near the onset of superconductivity, where $y \ll 1$ and y^3 can be neglected in Eq. (3). The position of the weak link, $\theta = 0$, is defined by $\int_0^{2\pi} D(\theta)\sin\theta\, d\theta = 0$ and $\gamma = -\int_0^{2\pi} D(\theta)\cos\theta\, d\theta > 0$. We found that no matter how small γ is, as long as it is positive, $y(0) = 0$ at the onset of superconductivity for $|K| = \frac{1}{2}$. The point $\xi = \xi_s$ at which this occurs is, to first order in the deviation of $D(\theta)$ from its average,

$$\xi_s \approx 2 + \gamma/\pi\,. \tag{7}$$

As ξ decreases (for $|K| = \frac{1}{2}$), it can be proven that $y(0)$ vanishes to at least first order in $\xi_s - \xi$. However, we still do not have a complete picture for the passage from the singly to the doubly connected phase, and the possibility of a gradual increase of $y(0)$ cannot be ruled out with certainty. As $\xi \to \xi_s^-$, if D has the form $D(\theta) = \sum_{i=0}^{\infty} D_i \sin^i(\theta/2)$, then $y(\theta) = \sum_{i=1}^{\infty} y_i \sin^i(\theta/2)$, with the recursion relation

$$y_i = \frac{1}{i(i-1)D_0} \sum_{j=1}^{i-1} y_j \{-j(i-1)D_{i-j} + [(i-2)j - 4/\xi_s^2]D_{i-j-2}\}, \tag{8}$$

with $D_{-1} = 0$ and y_1 a proportionality constant.

To summarize, we have found a stable topology of the order parameter (the SCP), which has not been previously discussed. Imperfections of the size considered in our calculations were most probably present in already performed experiments; these experiments looked only for the normal-superconducting transition, so that they could only detect the SCP through the shape of the transition line in the (T, H) plane. (T_s should not be influenced by the magnetic flux.) However, our numerical results indicate that the SCP exists only in a small region, and would not reveal itself unless one looks for it. (For comparison, the systematic measurements by Groff and Parks [1] provide the transition temperature at flux steps of the order of $\Delta K \gtrsim 0.03$; moreover, the temperature itself was not varied continuously, and there was thus no chance of hitting the $T = T_s$ line.) The SCP is characterized by the existence of a surface ($\theta = 0$) where the order parameter vanishes. This spatial dependence of the order parameter could be observed directly by means of scanning tunneling microscopy [4]. Also, the SCP has no supercurrent in the θ direction, and could be identified by means of a magnetometer.

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