Volume Operator in Discretized Quantum Gravity

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We investigate the spectral properties of the volume operator in quantum gravity in the framework of a previously introduced lattice discretization. The spectrum of the volume operator is discrete, but its eigenstates differ from those found in an earlier continuum treatment. This illustrates how lattice methods can be used profitably in the context of diffeomorphism-invariant theories.

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One of the most active branches of research into the quantization of (3 + 1)-dimensional gravity of the last few years has been the canonical, operator-based framework of the so-called loop approach. It is nonperturbative in the sense that it is not *a priori* restricted to the study of geometries close to flat Minkowski space. Its basic variables are (nonlocal) generalized Wilson loops of the $SL(2, \mathbb{C})$-valued Ashtekar connection. Also in the quantum theory, the state space and operators are labeled by (equivalence classes of) closed curves in three-space, which has led to considerable progress in solving the quantum constraints of the theory. The first, formal solutions to all of the constraints, including the Wheeler-DeWitt equation, were found in this loop formulation [1].

Although since then many of the mathematical ingredients of loop representations have been scrutinized and better understood (see, for example, [2]), one is still lacking a rigorous control over the regularization procedure necessary for obtaining a well-defined quantum Hamiltonian. One difficulty is the absence of a natural background metric in the "fully diffeomorphism-invariant phase" of the theory. Second, since the basic variables are nonlocal, the definition of the quantum Hamiltonian $\hat{H}$ usually involves a shrinking of loop operators to points, which arguably is a rather ill-defined process. These problems, and the absence of a well-defined scalar product in the quantum representation, have hampered progress toward a better understanding of the "solutions to all the constraints" and of observables.

In a recent paper [3], we have proposed an alternative regularization for the loop approach that does not involve a point splitting for the definition of the Hamiltonian constraint. It is a lattice regularization of the type used in quantum chromodynamics [4], but with two important differences. First, the lattice is considered as purely topological, and therefore the basic Wilson loop variables of the theory (with support on the links of the lattice) are both manifestly gauge and spatial diffeomorphism invariant. Second, since the "gauge group" $SL(2, \mathbb{C})$ is noncompact, we do not use the Haar measure to define the inner product, but a suitably defined measure on holomorphic $SL(2, \mathbb{C})$-valued wave functions, with respect to which the norm of holomorphic Wilson loop states is finite. Thus one may think of the construction as a finite approximation to the usual loop representation, where the support of loops has been restricted to a fixed cubic, topological lattice.

The main assets of the lattice model are its computational simplicity and the existence of a well-defined scalar product. In a preliminary investigation we were able to find a large number of solutions to the Wheeler-DeWitt equation that have finite norm with respect to this inner product. Furthermore, questions about the self-adjointness of operators, and in particular observables, can now be addressed. One test of this and other approaches is whether one can define physically interesting operators that are self-adjoint.

In this Letter, we will be concerned with the so-called volume operator, introduced in [5]. Although it is not an observable of the pure theory, there are arguments suggesting it will become one once matter has been included. Rovelli and Smolin have presented a partial computation of its spectrum, based on a certain continuum regularization [6]. According to their arguments, the spectrum is both real and discrete, which is taken by them to indicate a fundamental discreteness of the theory at the Planck scale. This result is formal in the sense that it postulates the existence of the quantum operators involved (and the limiting procedure used to define them), and of a scalar product that makes the spectrum calculation meaningful.

Given the scalar product of the lattice model, one may in turn ask whether an analog of the volume operator can be sensibly defined and whether its spectrum agrees with that found in the formal continuum calculation. We will show here that the answer to the first question is in the affirmative. The lattice regularizes in a natural way the terms cubic in momenta that appear in the definition of the volume operator, and its spectrum is again discrete. However, the nature of the eigenstates (to the extent they can be compared) disagrees with that found in [6]. In particular, we find that eigenstates of the volume operator are typically *complex* linear combinations of the Wilson loop states. We will also give a general argument for why trivalent spin network states are necessarily zero eigenvectors of the volume operator [7]. It is worth pointing out that spectral calculations within the lattice model are much simplified in comparison with the diagrammatic approach used in [6].

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The lattice formulation introduced in [3] takes place on a cubic $N \times N \times N$ lattice, with periodic boundary conditions. The basic operators associated with each lattice link l are a holomorphic SL(2, $\mathbb{C}$)-link holonomy $\hat{V}_A{}^B$ and a canonical momentum operator $\hat{p}_i$, with an adjoint index i. The wave functions are elements of the product space $\times_l L^2(\mathrm{SL}(2,\mathbb{C}), d\nu_t)^{\mathcal{H}}$. The measure is the heat kernel measure $d\nu_t$, and the superscript $\mathcal{H}$ denotes the subset of holomorphic L^2 functions. The basic commutators are

$$
\begin{aligned}
&[\hat{V}_A{}^B(n,\hat{a}), \hat{V}_C{}^D(m,\hat{b})] = 0\,,\\
&[\hat{p}_i(n,\hat{a}), \hat{V}_A{}^C(m,\hat{b})] = -\frac{i}{2}\,\delta_{nm}\delta_{\hat{a}\hat{b}}\,\tau_{iA}{}^B \hat{V}_B{}^C, \qquad (1)\\
&[\hat{p}_i(n,\hat{a}), \hat{p}_j(m,\hat{b})] = i\delta_{nm}\delta_{\hat{a}\hat{b}}\,\epsilon_{ijk}\,\hat{p}_k\,,
\end{aligned}
$$

where the links are labeled by their initial vertex n and a positive direction $\hat{a}$ emanating from it, and ϵ_{ijk} are the structure constants of SU(2). In terms of an explicit parametrization by $\alpha_i \in \mathbb{C}$, $i = 0, \ldots, 3$, $\sum_i \alpha_i^2 = 1$, the operators for a single link $(n, \hat{a})$ are given by

$$
\begin{aligned}
\hat{V}_A{}^B &= \begin{pmatrix} \alpha_0 + i\alpha_1 & \alpha_2 + i\alpha_3 \\ -\alpha_2 + i\alpha_3 & \alpha_0 - i\alpha_1 \end{pmatrix},\\
\hat{p}_1 &= \frac{i}{2}(\alpha_1\partial_0 - \alpha_0\partial_1 + \alpha_3\partial_2 - \alpha_2\partial_3),\\
\hat{p}_2 &= \frac{i}{2}(\alpha_2\partial_0 - \alpha_3\partial_1 - \alpha_0\partial_2 + \alpha_1\partial_3),\\
\hat{p}_3 &= \frac{i}{2}(\alpha_3\partial_0 + \alpha_2\partial_1 - \alpha_1\partial_2 - \alpha_0\partial_3). \qquad (2)
\end{aligned}
$$

Examples of SL(2, $\mathbb{C}$)-invariant states are the Wilson loops $\mathrm{Tr}V(\gamma) = \mathrm{Tr}V(l_1)V(l_2)\cdots V(l_n)$, where $\gamma = l_1 \circ l_2 \circ \cdots l_n$ is a closed lattice loop. Recall that we do not have an explicit coordinate expression for the heat kernel measure $d\nu_t$, and are therefore using the holomorphic transform C_t: $L^2(\mathrm{SU}(2), dg) \to L^2(\mathrm{SL}(2,\mathbb{C}), d\nu_t)^{\mathcal{H}}$ and its inverse to compute scalar products in $L^2(\mathrm{SL}(2,\mathbb{C}), d\nu_t)^{\mathcal{H}}$. It turns out that the operators $\hat{p}_i$ are self-adjoint in the holomorphic representation (the $\hat{V}_A{}^B$ are *not*); they are the holomorphic transforms of the corresponding self-adjoint differential operators on $L^2(\mathrm{SU}(2), dg)$.

The classical expression for the volume of a spatial region $\mathcal{R}$ is given by

$$
\begin{aligned}
\mathcal{V}(\mathcal{R}) &= \int_{\mathcal{R}} d^3x\,\sqrt{\det g}\\
&= \int_{\mathcal{R}} d^3x\,\sqrt{\frac{1}{3!}\,|\epsilon_{abc}\epsilon^{ijk}E_i^a E_j^b E_k^c|}\,, \qquad (3)
\end{aligned}
$$

where E_i^a are the momenta of the canonical Ashtekar variable pairs $(A_a^i(x), E_i^a(x))$. A natural discretization of the det$g$ term is $D(n) := \epsilon_{abc}\epsilon^{ijk}p_i(n,\hat{a})p_j(n,\hat{b})p_k(n,\hat{c})$, which in the continuum limit $a \to 0$ with respect to an arbitrary lattice spacing a goes over to $a^6\epsilon_{abc}\epsilon^{ijk} \times E_i^a E_j^b E_k^c + O(a^7)$. We may therefore take

$$
\mathcal{V}_{\mathrm{latt}} = \sum_{n\in\mathcal{R}} \sqrt{|\epsilon_{abc}\epsilon^{ijk}p_i(n,\hat{a})p_j(n,\hat{b})p_k(n,\hat{c})|}
$$

as the lattice analog of (3). The translation of this expression to the quantum theory is *a priori* not well defined, because of the presence of both the modulus and the square root. Fortunately, however, the operators

$$
\hat{D}(n) := \epsilon_{abc}\epsilon^{ijk}\hat{p}_i(n,\hat{a})\hat{p}_j(n,\hat{b})\hat{p}_k(n,\hat{c}) \qquad (4)
$$

are all self-adjoint, and hence we may go to a basis of $\times_l L^2(\mathrm{SL}(2,\mathbb{C}), d\nu_t)^{\mathcal{H}}$ consisting of simultaneous eigenfunctions of all the $\hat{D}(n)$ and *define* the operator $\hat{\mathcal{V}}_{\mathrm{latt}} = \sum_n \sqrt{|\hat{D}(n)|}$ through the square roots of the moduli of the eigenvalues of the $\hat{D}(n)$ in that basis.

There already exists a (partial) spectrum calculation in the continuum [6] we can compare with, obtained in terms of a basis of gauge- and spatially diffeomorphism-invariant states diagonal with respect to appropriate continuum analogs of the operators $\hat{D}(n)$ above. These states are given by so-called spin networks, constructed from trivalent (or n-valent) graphs whose edges are labeled by irreducible representations of SU(2), and vertices by SU(2)-intertwining operators (see, for example, [8] and references therein). They are certain (anti)symmetrized, real linear combinations of multiple Wilson loops with support on the graph. The difference with our discrete formulation is that one considers all possible graphs (with all possible labelings), whereas we keep the lattice fixed, and therefore the total number of degrees of freedom finite. Still, also the lattice approach allows for a similar construction of gauge-invariant states. Finding an efficient labeling for such states is a well-known problem in lattice gauge theory, and various methods have been used (see, for example, [9]). Typically such explicitly gauge-invariant bases are overcomplete, and for doing computations one needs an efficient way of labeling a complete subset of independent states. For spin network states of valence higher than three, one encounters a similar problem. Whether one basis is better than another is determined by the dynamics of the basic operators of the theory, and may be completely different for gravitational and gauge-theoretic applications.

When studying the volume operator it is indeed useful to consider a representation in which the lattice links are labeled by positive "occupation numbers," which count the number of (unoriented) flux lines of basic spin-$\frac{1}{2}$ representations on the link. This happens because the operators $\hat{D}(n)$ have a particularly simple structure: when acting on a multiple Wilson loop, they do not change its support (in terms of the flux line numbers), and only some finite-dimensional rearrangements occur within the subset of states that share the same occupation numbers.

The explicit part of the continuum spectrum calculation in [6] was made for trivalent spin networks. Although general gauge-invariant lattice states contain six-valent intersections, one can easily construct states that are only trivalent by assigning the occupation number zero to an appropriate subset of lattice links. The question therefore arises whether spin networks constructed from such trivalent states are also eigenstates in the lattice formulation. To answer it, it is sufficient to study the action of the operators $\hat{D}(n)$, as explained above.

We will now present the results of the spectral computation for small occupation numbers around a single vertex n, which will be sufficient to illustrate our point; a complete construction will appear elsewhere. The spectrum of $\hat{D}(n)$ is discrete. This was not clear *a priori*, since the group $SL(2, \mathbb{C})$ is noncompact; it is a consequence of our choice of a scalar product. We will not speculate here on whether this discreteness is of a fundamental nature or only an artifact of the regularization that will disappear in an appropriately taken continuum limit.

We will be interested in the behavior of gauge-invariant states under the action of $\hat{D}(n)$. One ingredient in the labeling of such a state is a 6-tuple $\vec{j}$ of integers $j_i \geq 0$ giving the occupation numbers $(j_1, \ldots, j_6)$ of the links $((n, \hat{1}), (n, \hat{2}), (n, \hat{3}); (n, -\hat{1}), (n, -\hat{2}), (n, -\hat{3})) \equiv ((n, \hat{1}), (n, \hat{2}), (n, \hat{3}); (n - \hat{1}, \hat{1}), (n - \hat{2}, \hat{2}), (n - \hat{3}, \hat{3}))$ intersecting at n (see Fig. 1). We will call $j := \sum_{i=1}^{6} j_i$ the *order* of a state (at n), which is an even integer. What remains to be specified is the way the j flux lines are joined pairwise at n to ensure gauge invariance. By convention we allow a flux line coming in from the positive $\hat{1}$ direction, say, to be joined only to a flux line from one of the other five links, and not from the same link (i.e., we forbid "retracings"). This leads to a constraint on the occupation numbers: any j_i has to be equal to or smaller than the sum of the remaining j_k, for example, $j_6 \leq \sum_{i=1}^{5} j_i$.

Given $\vec{j}$, the number of possible different contractions of flux lines at n is finite. Not all of them lead to linearly independent Wilson loop states: Consider a fixed (but arbitrary) extension of the flux line configuration, so as to obtain a set of closed lattice curves $\gamma_1, \gamma_2, \ldots, \gamma_k$ based at n. The corresponding multiple Wilson loop state is $\Psi = T_{\gamma_1} T_{\gamma_2}, \ldots, T_{\gamma_k}$, where we have abbreviated $T_\gamma := \mathrm{Tr} V(\gamma)$. Different contractions of the flux lines at n lead to different Wilson loop states (with the same support), which in general are related by so-called Mandelstam

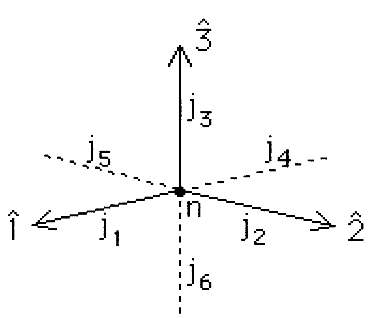

FIG. 1. Labeling of link directions meeting at a vertex n.

constraints. For example, for $k = 3$ one has [10]

$$T_{\gamma_1} T_{\gamma_2} T_{\gamma_3} = T_{\gamma_1} T_{\gamma_2 \circ \gamma_3} + T_{\gamma_2} T_{\gamma_1 \circ \gamma_3} + T_{\gamma_3} T_{\gamma_1 \circ \gamma_2} - T_{\gamma_1 \circ \gamma_2 \circ \gamma_3} - T_{\gamma_2 \circ \gamma_1 \circ \gamma_3} .$$

For the special case of a trivalent graph, Rovelli and Smolin have given a prescription for associating with each labeling of flux lines a *unique* quantum state, obtained by appropriately (anti)symmetrizing over all possible Wilson loop states sharing the same flux labels [6]: calling temporarily $j(l)$ the occupation number of a link l, the number of different multiloops one can associate with a given flux line labeling—by permuting the way individual flux lines are joined at vertices—is $\prod_l j(l)!$, where the product is taken over all lattice links. The spin network state is then obtained by adding the corresponding $\prod_l j(l)!$ Wilson loop states, with the weight $(-1)^{(p+n)}$, where p is the parity of the flux line permutation and n the number of closed loops in the multiloop.

The case of trivalent intersections turns out to be particularly simple, since there is only one way of contracting the (anti)symmetrized flux line configurations at each vertex. Following our earlier reasoning this means that $\hat{D}(n)\Psi = d\Psi$ for any trivalent spin network state Ψ on the lattice; i.e., Ψ is necessarily an eigenstate of $\hat{D}(n)$, with eigenvalue d.

Let us now compute the action of the operator $\hat{D}(n)$ on some trivalent lattice spin networks. The simplest type of configuration is of order 4 and has $\vec{j} = (2, 1, 1; 0, 0, 0)$. There are two possible permutations of the flux lines. The corresponding spin network state Ψ is the sum of the two, $\Psi = \psi_1 + \psi_2$. One finds $\hat{D}(n)\psi_1 = 0$ and $\hat{D}(n)\psi_2 = 0$, and therefore Ψ is a zero eigenvector.

At order $j = 6$ there are two admissible flux line labelings (up to a permutation of link labels). The first one is $\vec{j} = (2, 2, 2; 0, 0, 0)$, where there are $2!2!2! = 8$ flux line permutations, leading to Wilson loops states ψ_i, $i = 1, \ldots, 8$. One computes $\hat{D}(n)\psi_i = 0$, $\forall\, i$, and the spin network state, which is again the weighted sum of the ψ_i, is a zero eigenvector of $\hat{D}(n)$. Similarly, for the spin network state Ψ associated with $\vec{j} = (3, 2, 1; 0, 0, 0)$, one finds $\hat{D}(n)\Psi = 0$.

We will explain shortly why indeed $\hat{D}(n)\Psi = 0$ for *any* trivalent spin network Ψ. Before doing so, let us look at a couple of examples that lead to nontrivial eigenstates of $\hat{D}(n)$. First, consider $\vec{j} = (1, 1, 1; 1, 0, 0)$ (see Fig. 2; the dotted lines with arrows denote arbitrary extensions by other lattice links). The three possible Wilson loop states are $\psi_1 = \mathrm{Tr} V(\alpha) V(\beta)$, $\psi_2 = \mathrm{Tr} V(\alpha \circ \beta)$, and

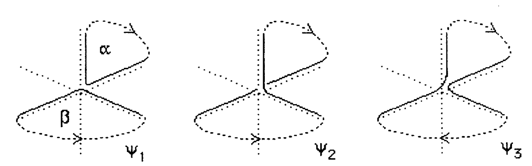

FIG. 2. Three possible contractions for $\vec{j} = (1, 1, 1; 1, 0, 0)$.

$\psi_3 = \mathrm{Tr}V(\alpha \circ \beta^{-1})$, where we have assigned a definite orientation to the composite loops α and β. The ψ_i are already spin networks in the sense that there are no flux line permutations to be taken into account. The action of $\hat{D}(n)$ yields

$$\hat{D}(n)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \frac{3i}{2}\begin{pmatrix}0 & -1 & 1\\ 2 & -1 & 0\\ -2 & 0 & 1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix},$$

and its eigenvalues are easily computed to be 0, $-3\sqrt{3}/2$, and $3\sqrt{3}/2$. The presence of a zero eigenvector Ψ is not surprising, since the three states ψ_i are not independent from the outset, but rather obey the Mandelstam constraint $\Psi = \psi_1 - \psi_2 - \psi_3 = 0$.

A nontrivial example of order 6 is given by $\vec{j} = (2,1,1;1,1,0)$. We only sketch the result: there are 12 different configurations to start with, from contracting the flux lines at n. Symmetrization with respect to the two flux lines in the $\hat{1}$ direction leaves us with 6 spin network states. After using the Mandelstam constraints [10], only three linearly independent spin network states remain. Diagonalizing the action of $\hat{D}(n)$ on those states, one finds the three eigenvalues 0, $-3\sqrt{2}$, and $3\sqrt{2}$.

The above examples show that it is possible to find eigenstates of spin network type whose eigenvalues are nonzero. However, in all cases where at a vertex n one can construct only a *single* spin network state [which therefore must be an eigenstate of $\hat{D}(n)$], its eigenvalue necessarily vanishes. This happens because the momenta $\hat{p}_i$ in our representation are represented self-adjointly, and according to (2) each contain a factor of i, so that $\hat{D}(n)$ is also proportional to i. It is, however, easy to see that $i^{-1}\hat{D}(n)$ maps a Wilson loop state $\mathrm{Tr}V(\gamma)$ into a *real* linear combination of such states. Therefore, if we have a spin network state Ψ (that by construction is a real linear combination of Wilson loop states), its eigenvalue equation is $\hat{D}(n)\Psi = d\Psi$, with *imaginary* d. On the other hand, $\hat{D}(n)$ is a self-adjoint operator and its eigenvalues are real. Hence we conclude that necessarily $d = 0$.

The above calculations took place around a single vertex n, but can be generalized immediately to lattice regions $\mathcal{R}$ containing several or even all of the lattice vertices, to obtain eigenstates of the volume operator $\hat{\mathcal{V}}_{\mathrm{latt}}(\mathcal{R})$. Although its spectrum is obviously discrete, we have found that all quantum spin network states corresponding to trivalent graphs are eigenstates with eigenvalue zero. This disagrees with the continuum computation of the trivalent sector reported by Rovelli and Smolin [6], where a nonvanishing spectrum was found.

The presence of factors of i in the definition of our momentum operators $\hat{p}_i$ can be traced back to the canonical commutators of the continuum Yang-Mills theory, $[\hat{A}_a^i(x), \hat{E}_j^b(y)] = i\delta_a^b\delta_j^i\delta^3(x-y)$, whose lattice analogs in the holomorphic representation are given by (1). However, we, strictly speaking, should be quantizing the classical Poisson brackets $\{A_a^i(x), E_j^b(y)\} = i\delta_a^b\delta_j^i\delta^3(x-y)$ of the canonical Ashtekar variables [11], leading to canonical commutators $[\hat{A}_a^i(x), \hat{E}_j^b(y)] = \delta_a^b\delta_j^i\delta^3(x-y)$ *without* a factor i. In [3], we quantized the commutators *with* the factor i, to facilitate comparison with the usual formalism of Hamiltonian lattice gauge theory. This was done with the understanding that the quantum commutators with and without i can in a straightforward way be related by multiplying the canonical momenta by i. For the case of our lattice variables, defining new momenta $\hat{p}_i' := i\hat{p}_i$ leads to a version of the basic commutator algebra (1) without any factors of i on the right-hand sides. The substitution $\hat{p}_i \longrightarrow \hat{p}_i'$ does not change the main results obtained above: the operators $\hat{p}_i'$ and the corresponding composite operators $\hat{D}(n)'$ become anti-Hermitian, and their spectra purely imaginary. Otherwise, the spectra remain discrete, and trivalent spin network states are still eigenstates with zero eigenvalues.

The differences between the continuum and lattice regularizations make a direct comparison of the spectral computations difficult. Also it is not *a priori* clear to what extent they should agree, given that no continuum limit has yet been performed in the lattice formulation. Even if eventually an agreement on the vanishing of $\hat{\mathcal{V}}_{\mathrm{latt}}$ on trivalent states can be reached, it does not automatically follow that the nonzero parts of the spectra will coincide in both formalisms. The fact that in our approach the trivalent spin network states all "have zero volume" may be taken as an indication that they are degenerate from a physical point of view, and that it is not sufficient to consider trivalent states only.

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