Properties of Consistent Histories

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(Received 26 August 1994; revised manuscript received 21 April 1995)

We describe some properties of consistent sets of histories in the Gell-Mann–Hartle formalism, and give an example to illustrate that one cannot recover the standard predictions, retrodictions, and inferences of quasiclassical physics using the criterion of consistency alone.

PACS numbers: 03.65.Bz

Standard quantum theory, in the Copenhagen interpretation, gives a robust and successful algorithm for predicting the results of laboratory experiments. (For definiteness consider the version of the Copenhagen interpretation set out by Landau and Lifschitz [1].) It has, however, nothing to say about the larger quantum universe outside the laboratory, since its subject is solely the results of measurements made by some classical measuring apparatus whose existence is taken as an *a priori* assumption. New theories which make predictions without this last assumption would be of great interest, since they would have greater predictive power.

The consistent histories approach of Griffiths, Omnès, Gell-Mann, and Hartle has been suggested to be just such a development, extending the Copenhagen interpretation [2–5] and allowing us to make predictions in quantum cosmology where the quantum system is the whole Universe [6–8]. It is a formalism from which, it is hoped, the largely classical world of our experience might be *deduced,* rather than assumed. Thus, our observations of large-scale classical structure in the Universe, of macroscopic objects following classical equations of motion, and of definite classical outcomes to quantum experiments, are all supposed to be predictions, unconditionally derivable from the formalism. If these hopes were to be realized, the consistent histories approach would indeed have provided a significant increase in our predictive power. They rest, however, on as yet incomplete interpretational arguments and have naturally led to much debate [9,10]. Our own arguments, together with a critique of the existing literature, will be given in detail elsewhere [11]. Our aim here is to draw attention to some perhaps counterintuitive properties of consistent sets of histories, most of which have not previously been discussed in any detail in the consistent histories literature, and to explain their physical relevance.

The consistent histories formalism.—We begin with a brief description of the consistent histories formalism as it applies to nonrelativistic quantum mechanics, in the Heisenberg picture, using the language of projection operators and density matrices. We assume that a Hilbert space $\mathcal{H}$ and Hamiltonian H are given, that Hermitian operators correspond to observables, that the commutation relations among the Hamiltonian and physically interesting observables (such as position, momentum, and spin) have been specified, and that the operators corresponding to the same observables at different times are related by

$$P(t) = \exp(iHt/\hbar)P(0)\exp(-iHt/\hbar)\,. \qquad (1)$$

We are interested in a system (in principle, the Universe) whose initial density matrix ρ_i is given. We require that ρ_i is positive semidefinite. The formalism also allows a final condition to be imposed, an interesting generalization of standard quantum theory, though we shall not consider that possibility explicitly here. The basic physical events we are interested in correspond to sets σ of orthogonal Hermitian projections $P^{(i)}$, with

$$\sum_i P^{(i)} = 1 \text{ and } P^{(i)}P^{(j)} = \delta_{ij}P^{(i)}. \qquad (2)$$

These projective decompositions of the identity are applied at definite times, which we append to the sets of projections: thus $\sigma_j(t_j) = \{P_j^{(i)}; i = 1, 2, \ldots, n_j\}_{t_j}$ defines a set of projections obeying (2) and applied at time t_j. However, since our results depend only on the time ordering, we will generally omit explicit time labels. Suppose now we have a list of sets $\sigma_j(t_j)$ of this form, with j running from 1 to n, at times t_j with $t_i < t_1 < \cdots < t_n < t_f$. Then the histories given by choosing one projection from each σ_j in all possible ways are an exhaustive and exclusive set of alternatives S. We use Gell-Mann and Hartle's decoherence condition, and say that the histories form a *consistent set* if

$$\mathrm{Tr}(P_n^{(i_n)}, \ldots, P_1^{(i_1)}\rho_i P_n^{(j_1)}, \ldots, P_n^{(j_n)}) = \delta_{i_1 j_1}, \ldots, \delta_{i_n j_n} \times p(i_1, \ldots, i_n)\,, \qquad (3)$$

in which case $p(i_1, \ldots, i_n)$ is the probability of the history $P_1^{(i_1)}, \ldots, P_n^{(i_n)}$. [Gell-Mann and Hartle term is a set satisfying (3) *medium decoherent.*]

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We say the set

$$S' = (\rho_i, \{\sigma_1, \dots, \sigma_k, \tau, \sigma_{k+1}, \dots, \sigma_n\}) \qquad (4)$$

is a *consistent extension* of a consistent set of histories $S = (\rho_i, \{\sigma_1, \dots, \sigma_n\})$ by the set of projections $\tau = \{Q^i : i = 1, \dots, m\}$ if τ satisfies (2) and S' is itself consistent. We say the consistent extension S' is *trivial* if, for each history $\{P_1^{(i_1)}, \dots, P_k^{(i_k)}, P_{k+1}^{(i_{k+1})}, \dots, P_n^{(i_n)}\}$ in S, at most one of the extended histories $\{P_1^{(i_1)}, \dots, P_k^{(i_k)}, Q^i, P_{k+1}^{(i_{k+1})}, \dots, P_n^{(i_n)}\}$ has nonzero probability. We extend these definitions by taking consistent extension and trivial consistent extension to be transitive relations.

Counting consistent sets.—Let us now take the Hilbert space $\mathcal{H}$ to be of finite dimension N and ask: How many consistent sets are there? We first describe how consistent sets may be classified. The basic objects in the formalism are the projective decompositions of the identity $\sigma_j = \{P_j^{(i)} : i = 1, 2, \dots, n_j\}$, where the $P_j^{(i)}$ satisfy (2). These decompositions are parametrized by the set of ranks $\{r_j^{(1)}, r_j^{(2)}, \dots, r_j^{(n_j)}\}$ of the projection operators, where $N = \sum_{i=1}^{n_j} r_j^{(i)}$ and we take $r_j^{(1)} \geq r_j^{(2)} \geq \cdots r_j^{(n_j)}$, and the manifold

$$G(N; r_j^{(1)}, r_j^{(2)}, \dots, r_j^{(n_j)}) = \frac{U(N)}{[U(r_j^{(1)}) \times U(r_j^{(2)}) \times \cdots \times U(r_j^{(n_j)})] \times J}. \qquad (5)$$

J is a discrete symmetry group that eliminates overcounting when some of the ranks are equal.

The parameter space of a set of histories is then a manifold M, which is a product of such G's, one for each projective decomposition σ_k, $k = 1, 2, \dots, n$ (i.e., one for each time, t_k): $M = G_1 \times \cdots \times G_n$.

It is easy to use this parametrization in explicit calculations: One can define projections $\{P^{(1)}, \dots, P^{(n_j)}\}$ of ranks $\{r^{(1)}, r^{(2)}, \dots, r^{(n_j)}\}$ by choosing an orthonormal basis of vectors $\{x_1, \dots, x_n\}$, so that

$$P^{(1)} = \sum_{i=1}^{r_1} x_i (x_i)^\dagger, \quad P^{(2)} = \sum_{i=r_1+1}^{r_1+r_2} x_i (x_i)^\dagger, \qquad (6)$$

and so on. The redundancies in this parametrization correspond to the actions of the quotient subgroups, and can be eliminated at any convenient point. This can be done each time. Thus, in principle, we can simply fix the form of the initial density matrix, fix the ranks of the projection operators in the type of consistent set we wish to classify, and then impose the consistency conditions (3). These will be algebraic equations defining a submanifold L, the submanifold of consistent sets of the manifold of all sets of histories M.

While these algebraic equations are generally very complicated, one can at least make educated guesses about the qualitative features of L, such as its dimension, and these guesses can be checked in simple examples. A typical physical illustration of the consistent histories formalism would involve a small number of projection operators, describing quasiclassical operators in a large Hilbert space. One might, for instance, describe a coarse-grained trajectory of a dust grain, interacting with a photon background. Such physical projection operators rarely form a precisely consistent set, and there has been debate over whether or not it is possible to find close approximations to the projection operators which are exactly consistent. Comparing the number of parameters used to specify sets in M (very large) with the number of consistency equations (rather small) suggests that this is generically possible. If so, there is no need to follow Gell-Mann and Hartle in ascribing a fundamental role to approximately consistent sets: exactly consistent sets suffice. Moreover, the counting arguments show that in any physically realistic situation the dimension of L, the space of consistent sets, is very large.

Properties of consistent sets.—So, let us suppose that physics is described by exactly consistent sets and look at what this implies. We omit proofs, which can be found in Ref. [11].

Lemma 1.—Let $S = (\rho, \{\sigma_1, \dots, \sigma_k\})$ be a consistent set which is not a trivial extension of any consistent subset, defined on a space $\mathcal{H}$ of dimension N, with initial density matrix ρ of rank r. Then the length k of S obeys $k \leq rN$. (In particular, if ρ is pure then $k \leq N$.) (A similar result has been obtained independently by Diósi [12].)

In other words, if the Hilbert space of the Universe is finite dimensional, there is a strict bound on the number of probabilistic physical events. Once this number has occurred, the evolution of the Universe continues completely deterministically. This is mathematically an unsurprising feature of the formalism but, as far as we are aware, physically quite new: No previous interpretation of quantum theory has suggested that quantum stochasticity is exhaustible in this way.

In the consistent histories approach, predictions can only be made once a consistent set—*the* physically relevant set—is fixed. The key problem in interpreting the formalism is explaining how, given the profligate abundance of consistent sets, this is to be done. Once the choice has been made, one can simply declare by fiat that physics should be described by one history from the relevant set, chosen at random using the decoherence functional probability distribution. Again, the key question is whether the choice has been made *within* the formalism or whether it relies on assumptions that go beyond it.

It thus becomes an important question whether, when some of the projective decompositions in the relevant set are known, others can be determined. In particular, if, taking the past and present for granted, we were able to deduce the form of the relevant set in the future using only consistency criteria, then we could indeed make unconditional predictions about the future within the consistent histories formalism: The choice of set would be clearly determined by the formalism. This, though, is generally false.

Lemma 2.—Let S be a set of consistent histories for which there exists a nontrivial consistent extension. Let S have a pure initial state ρ, and let $\mathcal{H}$ be either finite dimensional or separable. Then there exists a continuous family of nontrivial extensions for each history in S with nonzero probability.

So, if a consistent set describing a physical system up to time t leaves some future events unpredictable, there are infinitely many different consistent continuations of that set. In particular, if a consistent set describes, in Gell-Mann and Hartle's language, quasiclassical physics—involving operators describing the same types of variables at different times, following largely deterministic evolution equations—up to time t then, if any unpredictability remains, almost all future consistent continuations will *not* be quasiclassical. Whatever our experience of a persistently quasiclassical world may be ascribed to—and there are various suggestions [11]—it does not follow simply from consistency.

Still, one might hope that at least, if the setup to time t is quasiclassical, then any nonquasiclassical consistent future extension can consistently incorporate future deterministic quasiclassical predictions. Indeed, Omnès has suggested that this is so [5]. But in fact, as Omnès now accepts, this fails quite generally. If unpredictability remains, then there are no future predictions consistent with all consistent extensions of the present data.

Lemma 3.—Let $S = (|\psi\rangle\langle\psi|, \{\sigma_1, \dots, \sigma_l\})$ be as in Lemma 2, with $\mathcal{H}$ finite dimensional or separable. Then there is no projective decomposition σ_{l+1} such that (i) $S' = (\rho, \{\sigma_1, \dots, \sigma_l, \sigma_{l+1}\})$ is a consistent extension of S and (ii) any consistent extension $S'' = (\rho, \{\sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r\})$ of S has a consistent extension $(\rho, \{\sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_r, \sigma_{l+1}\})$.

We have only been able to identify one class of statements which *can* consistently be added to any consistent extension of a set. These arise where the set contains the same decomposition twice. In this case, further repetitions can be included, provided that they are made between the first two.

Lemma 4.—Let $S = (\rho, \{s_1, \dots, s_j, t, t_1, \dots, t_l, t, s_{j+1}, \dots, s_k\}) \equiv (\rho, \{S_1, t, T, t, S_2\})$ be a consistent set in which the projective decomposition t is repeated. Let $S' = (\rho, \{S_1, t, T_1, t, T_2, t, S_2\})$ be an extension of S by a further repetition of t at some point between the first two, so that $\{T\} = \{T_1, T_2\}$. Then S' is also consistent.

Put picturesquely, if a tree is observed standing in the forest at dusk and dawn, and if the dynamics cause no qualitative complications, then the formalism allows us unambiguously to deduce that it remained standing overnight while unobserved.

A simple example illustrates the weaknesses of the consistency criterion. Consider two systems, described by two-dimensional Hilbert spaces V and W, with orthonormal bases $\{v_1, v_2\}$ and $\{w_1, w_2\}$. We suppose that the total Hamiltonian is zero except between times $t = t_1$ and $t = t_2$, during which the systems are coupled by an interaction which models a measurement process. Specifically, we take the unitary evolution operator between these times to be the operator U defined by

$$U|v_1\rangle|w_1\rangle = |v_1\rangle|w_1\rangle, \quad U|v_2\rangle|w_1\rangle = |v_2\rangle|w_1\rangle,$$
$$U|v_1\rangle|w_2\rangle = |v_2\rangle|w_2\rangle, \quad U|v_2\rangle|w_2\rangle = -|v_1\rangle|w_2\rangle, \tag{7}$$

and we take $\rho_i = |v_1\rangle\langle v_1| \otimes I$. In this much simplified (and unrealistic) model, V represents the relevant degrees of freedom—two "pointer positions"—of a measuring device examining a two-dimensional microscopic quantum system represented by W. We have chosen ρ_i so that the initial pointer position is specified and no information is known about the system W, which might, for example, be a spin-$\frac{1}{2}$ cosmic ray.

Now consider an experiment in which the pointer is observed to be in state $|v_1\rangle$ at time $t_3 > t_2$, so that the combined system lies in the range of the projection $P = |v_1\rangle\langle v_1| \otimes I$. The standard description of this experiment would distinguish between the macroscopic system V, which follows classical dynamics after t_2 and the microscopic system W, which is observed at time t_3 to be in the state $|w_1\rangle$ and thereafter follows the Schrödinger equation. Now the consistent histories description of the observation uses the set S defined by the single projective decomposition $\{P, 1 - P\}$ at time t_3, and specifically the history from that set defined by P.

However, none of the standard inferences drawn from the observation can be made using the consistency criterion alone: We cannot deduce that the pointer was in state $|v_1\rangle$ at times between t_2 and t_3 or after time t_3 (as classical mechanics would imply), nor that the system was in state $|w_1\rangle$ after t_3 (as standard quantum mechanics would imply). The reason is that in each case we can find a consistent extension of S with which the inference is inconsistent.

For example, let Q be the projection $I \otimes |w\rangle\langle w|$ onto the state $|w\rangle = (|w_1\rangle + |w_2\rangle)/\sqrt{2}$ and R the projection $I \otimes |w_1\rangle\langle w_1|$. Then the set $\{\{P, 1 - P\}(t_3), \{Q, 1 - Q\}(t)\}$ is consistent. (We now include explicit time labels and take $t_3 < t < t'$.) However, since the extended set $\{\{P, 1 - P\}(t_3), \{Q, 1 - Q\}(t), \{R 1 - R\}(t')\}$ is inconsistent, we cannot infer that the quantum mechanical system W is in state $|w_1\rangle$ at any time $t' > t_3$. Essentially the same argument shows that this inference cannot be made at time t_3. In other words, the observation of a pointer state does not imply the result of its measurement in the usual way. Similar arguments show that we cannot infer that the pointer itself is in state $|v_1\rangle$ at any time other than t_3.

In conclusion, these results illustrate interesting features of the consistent histories formalism. Those who prefer their fundamental theories to be mathematically precise will be encouraged that the use of approximately consistent sets can apparently be avoided. On the other hand, it will be seen that consistency itself is a very weak condition: There is generally a large variety of consistent sets. We stress that this does not mean that the consistency criterion is incompatible with experiment. Many careful studies have illustrated the efficiency of the decoherence process and its crucial importance in understanding the dynamics of quasiclassical systems. (See, for example, the decoherence calculations of Ref. [13].) Assuming that quantum theory holds good as a description of macroscopic systems, the moral to be drawn from these studies is that, in any realistic description of an experiment, one of the consistent sets will correctly describe familiar quasiclassical physics and will allow the standard predictions and inferences.

The problem here is that all consistent sets have the same status in the formalism, most have very little to do with the quasiclassical world of our observations, and we can make the predictions we would like to make only after we have made the choice of one particular set: the familiar quasiclassical one. If the consistent histories formalism is to represent an enhancement in predictive power over the Copenhagen interpretation, it requires intepretational arguments that show this choice to be determined by the formalism. Omnès' interesting attempt [4,5] to find a mathematical criterion that correctly identifies the relevant set, unfortunately, fails. The remaining arguments in the literature [6,7] which suggest that the formalism nonetheless does explain the observed persistence of quasiclassicality therefore deserve careful scrutiny. Our own conclusion [11] is that they rely on important hidden assumptions.

We thank the Aspen Institute for their hospitality in the formative stages of this work. F. D. thanks, in turn, the Newton Institute in Cambridge for their hospitality during part of this work. F. D. was supported by NSF Grant No. PHY-9008502; A. K. by a Royal Society University Research Fellowship. It is a pleasure to thank Murray Gell-Mann, Robert Griffiths, and Roland Omnès for invaluable discussions of their work. We are particularly grateful to Jim Hartle for many helpful and stimulating discussions.

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