Comment on "Reduced Dynamics Need Not Be Completely Positive"

Recently, in his Letter, Pechukas [1] has challenged the widely used formalism in the theory of open quantum systems, namely, the completely positive dynamical maps [2]. The aim of this Comment is to show that the complete positivity of the reduced dynamics should be and can be preserved.

The generally accepted scheme for the reduced dynamics is the following. The dynamical map for an open system S interacting with a reservoir R is given by

$$\rho_S \mapsto \Lambda\rho_S = \mathrm{tr}_R(U\Phi\rho_S U^*), \tag{1}$$

where Φ is a general "assignment map" which assigns to each ρ_S a single state of the total system $S + R$. U is a unitary dynamics of the composed system. Obviously, Λ is a composition of Φ and two completely positive maps (the unitary evolution and the partial trace map), and hence only the properties of Φ are under discussion. Impose the following "natural" conditions on Φ: (a) Φ preserves mixtures, (b) Φ is consistent in the sense that $\mathrm{tr}_R\Phi\rho_S = \rho_S$, and (c) $\Phi\rho_S$ is positive for all positive ρ_S. Pechukas proved for the case of two-dimensional Hilbert space that the only maps Φ which satisfy all the conditions (a), (b), and (c) for all initial density matrices are product maps $\Phi\rho_S = \rho_S \otimes \omega_R$ (with ω_R being a fixed state of R) [3] and rightfully stressed that beyond the weak coupling regime the product initial condition is inappropriate. His proposal is to give up the positivity condition (c) as it is the case for the following assignment used in the recent literature [4]:

$$\Phi\rho_S = (\rho_S {\rho_S^{\mathrm{eq}}}^{-1} \rho_{SR}^{\mathrm{eq}} + \rho_{SR}^{\mathrm{eq}} {\rho_S^{\mathrm{eq}}}^{-1} \rho_S)/2. \tag{2}$$

Here ρ_{SR}^{eq} represents the equilibrium state of the composed system and $\rho_S^{\mathrm{eq}} = \mathrm{tr}_R\rho_{SR}^{\mathrm{eq}}$. Obviously Eq. (2) satisfies conditions (a) and (b), but not (c). To avoid negative probabilities Pechukas proposed to restrict ourselves to such initial density matrices for which $\Phi\rho_S \geq 0$. Unfortunately, it is impossible to specify such a domain of positivity for a general case, and moreover there exists no physical motivation in terms of operational prescription which could lead to the assignment [Eq. (2)]. The point of this Comment is to propose a mathematically consistent scheme with a clear operational meaning for the reduced dynamics. Let us introduce a completely positive map T ("operation") $\rho_S \mapsto T\rho_S = \sum_n V_n\rho_S V_n^*$, $\sum_n V_n^* V_n = I$ (a sum can be replaced by an integral) which represents the influence of a certain instrument preparing the state, and define the associated assignment map as

$$\Phi\rho_S = \sum_n V_n\rho_S V_n^* \otimes \mathrm{tr}_S(V_n^* V_n \rho_{SR}^{\mathrm{eq}})/\mathrm{tr}(V_n^* V_n \rho_{SR}^{\mathrm{eq}}). \tag{3}$$

Obviously Φ given by Eq. (3) satisfies conditions (a) and (c) (Φ is completely positive). This map fails to satisfy (b) for a general initial ρ_S, but the domain of validity of (b) is easily identified as density matrices which are invariant under T, i.e., $T\rho_S = \rho_S$. For other initial conditions the "inconsistency" of the map Φ reflects the instantaneous perturbation of the state of S due to the preparation and measurement process.

The first example of T corresponds to an ideal measurement of an observable A with a discrete spectrum, i.e., $A = \sum_n a_n P_n$, where P_n are projectors. Then, putting $V_n = P_n$, we obtain Φ which satisfies (b) for all ρ_S commuting with A. In the second example we define T as an integral $T\rho_S = \int \mu(d\Gamma_S)|\Gamma_S\rangle\langle\Gamma_S|\rho_S|\Gamma_S\rangle\langle\Gamma_S|$, where $\{|\Gamma_S\rangle\}$ is an overcomplete set of coherent vectors. For all kinds of coherent vectors and sufficiently smooth ρ_S, $T\rho_S \mapsto \rho_S$ in the semiclassical limit.

There exists another possibility, namely, to give up the condition (a) rather than (b) by adapting the preparation process to a given initial $\rho_S = \sum_n \lambda_n P_n$. Then we construct the assignment map as in the first example, using a state-dependent, completely positive map T with $V_n = P_n$. Hence $\Phi\rho_S = \sum_n \lambda_n P_n \otimes \mathrm{tr}_S(\rho_{SR}^{\mathrm{eq}} P_n)/\mathrm{tr}(\rho_{SR}^{\mathrm{eq}} P_n)$. This map satisfies conditions (b) and (c) but is not linear except in the weak coupling regime where $\rho_{SR}^{\mathrm{eq}} = \rho_S^{\mathrm{eq}} \otimes \rho_R^{\mathrm{eq}}$, and we recover the product assignment map.

In conclusion, one should stress that beyond the weak coupling regime there exists no unique definition of the quantum reduced dynamics. It is due to the fact that any physical process of preparation of the initial state of S disturbs the state of R as well. Choosing mathematical models of the preparation process (operation T), one can define consistently various assignment maps and hence various reduced dynamics. All of them satisfy the fundamental positivity condition and can be expressed in terms of completely positive maps.

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