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Array Enhanced Stochastic Resonance and Spatiotemporal Synchronization

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We enhance the response of a "stochastic resonator" by coupling it into a chain of identical resonators. Specifically, we show via numerical simulation that local linear coupling of overdamped nonlinear oscillators significantly enhances the signal-to-noise ratio of the response of a single oscillator to a time-periodic signal and noise. We relate this *array enhanced stochastic resonance* to the global spatiotemporal dynamics of the array and show how noise, coupling, and bistable potential cooperate to organize spatial order, temporal periodicity, and peak signal-to-noise ratio.

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A noisy nonlinear system exhibits *stochastic resonance* (SR) if its response to a deterministic signal is optimized by a nonzero value of the noise [1,2]. Over the past decade this phenomenon has generated considerable interest [3], and recent results indicating the possibility of an enhancement of the effect in globally coupled arrays with linear [4] or nonlinear [5,6] couplings have fueled speculation regarding its utility in signal processing and device applications. Recent theoretical results [7] afford a glimpse into the richness of behavior that is possible in large arrays of noisy coupled oscillators. In this Letter, we show that *local, linear* coupling can enhance SR in a chain of nonlinear oscillators. We understand this enhancement in terms of the collective spatial *and* temporal motion of the array.

The phenomenon of SR has universal flavor, having been demonstrated in a wide range of systems [3]. For isolated nonlinear elements, conventional SR enables the use of noise as a design parameter. For arrays of nonlinear elements, *array enhanced SR* (AESR) suggests an additional design parameter: the coupling strength. We believe AESR, along with its attendant synchronization phenomena, is a novel use of noise which may be significant in biological systems and which may be important for engineers, especially in situations where the possibilities of linear systems have been exhausted, or where operating conditions are extreme.

In order to demonstrate noise-induced synchronization and AESR most generally, we consider as simple a model system as possible. We study a one-dimensional array (or chain) of damped driven nonlinear oscillators

$$m\ddot{x} + \gamma\dot{x} = kx - k'x^3 + A\sin\omega t\,. \qquad (1)$$

We choose $k, k' > 0$ to ensure a bistable potential. To reduce the dimension of the parameter space, we study the overdamped limit $m\ddot{x} \ll \gamma\dot{x}$ and neglect the inertial term with respect to the viscous term. (However, AESR occurs even if the inertial term is retained [8].) We couple these bistable elements linearly to their nearest neighbors and employ free boundary conditions. A typical overdamped oscillator n evolves according to

$$\dot{x}_n = kx_n - k'x_n^3 + A\sin\omega t + \varepsilon(x_{n-1} - x_n) + \varepsilon(x_{n+1} - x_n) + N_n(t)\,. \qquad (2)$$

We imagine $N_n(t)$ to be Gaussian white noise. However, in practice, $N_n(t)$ is band limited with a (one-sided) spectrum of height $2D$ out to a very high frequency f_N and zero beyond. We characterize the noise by its mean squared amplitude or *noise power* $\sigma^2 = 2Df_N$. We emphasize the case of incoherent or *local noise,* where the noise is uncorrelated from site to site, as opposed to the case of coherent or *global noise,* where the noise is identical at each site.

We numerically integrate the stochastic differential equation (2) using the Euler-Maruyama scheme [9] with a time step $dt = 1/(2f_N)$. We use the time series of a single oscillator in the array to compute a power spectral density (PSD). We typically average four PSD segments, each of 32 forcing periods per segment, and 4096 samples (time steps) per forcing period. This centers the forcing frequency on bin 32 of the PSD, which we call the signal bin. To reduce bin leakage we first convolve the signal with a Welch window.

We characterize SR by a single-to-noise ratio, defined here as the ratio of the signal power divided by the noise power in the signal bin, expressed in dB. The noise power is estimated by performing a nonlinear fit by the PSD around, but not including, the signal bin. The signal power is estimated by subtracting this noise background from the total power in the signal bin, taking into account that Welch windowing a long time series scales narrow-band peaks by a "processing gain" of about 0.83 [2]. Thus,

$$\mathrm{SNR} = 10\log_{10}\left[\frac{(\text{total power} - \text{noise power})/\text{gain}}{\text{noise power}}\right]. \tag{3}$$

Variations of this definition do not qualitatively alter our results.

Since we are often interested in binary signals and quantized-output systems (e.g., neurons and some charged couple device arrays), we filter the unquantized analog time series to generate a binary output of ± 1, reflecting which well the oscillator is in. To remove insignificant excursions across the potential barrier, we give the filter a small hysteresis; however, this does not qualitatively alter our results. Furthermore, we emphasize that *both the SR peak and its enhancement in coupled arrays are clearly evident even in the unfiltered (i.e., analog) SNR curves.*

We choose our operating regime just below the deterministic switching threshold so that in the absence of noise the oscillator is confined to a single well of the bistable potential, but small noise can induce significant hopping between wells. However, we observe similar phenomena even at substantially lower forcing amplitudes [10]. The operating parameters $k = 2.1078$, $k' = 1.4706$, $A = 1.3039$, and $f = \omega/2\pi = 0.116$ (used throughout this work) are not otherwise special.

Figure 1 illustrates how the SNR curve changes as a single resonator is coupled more and more tightly into the middle of an array of identical resonators. As the noise increases, the SNR peak rises, shifts to higher noises, and then subsides. With increasing coupling strength it becomes increasingly difficult for the noise to kick an oscillator, along with its coupled neighbors, over the barrier. Thus the binary SNR curve shifts to higher noises. The SR peak in the analog curve rises as the coupling strength increases from zero to two. Heuristically, coupling favors coherent motion over incoherent motion.

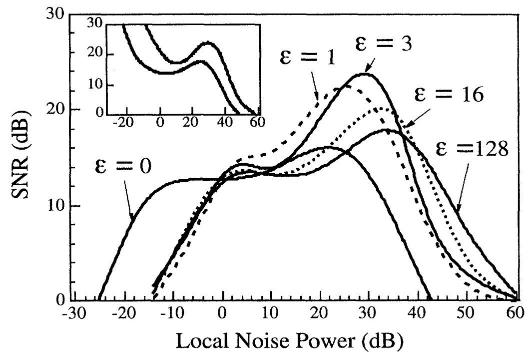

FIG. 1. SNR output of the middle of nine oscillators. Curves rise, shift, and subside as coupling increases. Numerical uncertainty, arising from averaging PSD's of noisy time series, is ± 0.5 dB. Cubic splines have been fit to the data to aid the eye. Inset: Unfiltered analog curves rise as coupling increases from $\varepsilon = 0$ (lower curve) to $\varepsilon = 2$ (upper curve). Parameter values are $k = 2.1078$, $k' = 1.4706$, $A = 1.3039$, and $f = \omega/2\pi = 0.1162$.

Figure 2 illustrates how the SNR, maximized over noise, varies with coupling strength and chain length. In the limit of low coupling, the oscillators are independent and behave as if isolated. In the limit of high coupling, the oscillators are rigidly connected and behave as a single oscillator. In between, the coupling enhances the coherence of the oscillators. Even for a chain of length 9, the SNR enhancement is 6 dB.

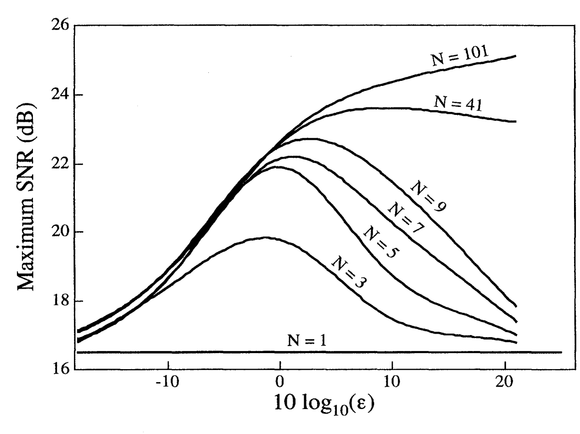

FIG. 2. SNR of middle oscillator, maximized over (local) noise, plotted as a function of coupling for various chain lengths. Numerical uncertainty is ± 0.5 dB. Cubic splines have been fit to the data to aid the eye. Other parameter values as in Fig. 1.

At low to moderate coupling, the SNR of the middle oscillator exceeds the SNR of the end oscillators. However, at large coupling the SNR's of the middle and end oscillators are indistinguishable to within numerical uncertainty. Large coupling homogenizes the response of the chain.

We can relate the global dynamics of the array to the local behavior of component oscillators. The sequence in Fig. 3 reveals the spatiotemporal behavior of the chains at successively higher noises. Each sequence displays the evolution of a chain of 101 oscillators, time increasing upward. Black corresponds to one well and white to the other. The oscillators in the chain all begin in the black well, but with a random distribution of positions. (In Fig. 3, a short transient has been deleted.) As the chains evolve, distinct collective behaviors become apparent at different noise levels, and these correspond to features in the individual SNR curves. Low noise rarely forces an oscillator in the chain into the opposite well. Consequently, the chain as shown is mainly confined to the black well, except for small segments that make minor excursions into the white well. The binary SNR's of the component oscillators are close to zero. Higher noise begins to move the chain, and large segments switch wells. The domains of black and white segments expand and contract as oscillators tug their coupled neighbors back and forth, and the binary SNR's rise. Still higher noise (20 dB at $\varepsilon = 1/4$, 30 dB at $\varepsilon = 16$) maximizes the degree of entrainment between noise and signal. The chain is synchronously forced between the two potential wells, and the oscillator SNR's are maximized. Greater noise overwhelms the coupling, forcing different parts of the chain to become uncorrelated. Adjacent oscillators occupy opposite wells, and the SNR's drop. The behavior of chains of 512 oscillator is very similar [10].

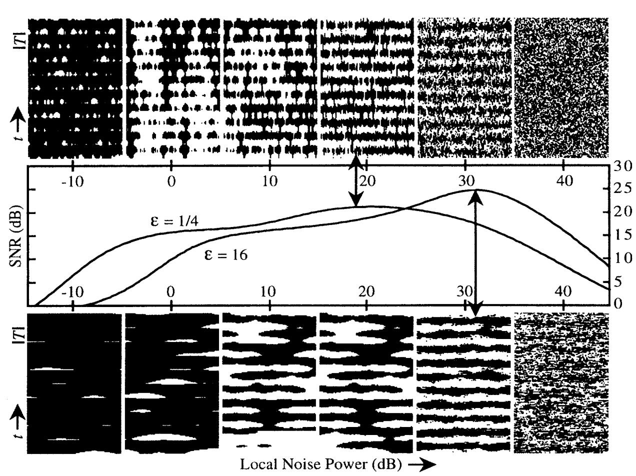

FIG. 3. Sequence of spatiotemporal dynamics of a 101 oscillator array. In each frame, the array is horizontal and time increases upward ($T = 2\pi/\omega$). One well of the bistable potential is colored black, the other white. Local noise power increases from left (-10 dB) to right (40 dB) in 10 dB steps. Noise-induced synchronization corresponds to maximum SNR: 20 dB at $\varepsilon = 1/4$ (top sequence) and 30 dB at $\varepsilon = 16$ (bottom sequence). Other parameter values are as in Fig. 1.

When the noise is adjusted to maximize the SNR of an individual oscillator, there is both spatial organization and temporal periodicity in the chain as a whole, as is clearly seen in Fig. 3. One might have expected disorganization in the presence of such a large noise. However, the coupling, in cooperation with the noise and the bistable potential, organizes the chain in space and time. This spatiotemporal organization of the array can be quantified by the *occupancy* function. Noting that the periodic forcing effectively rocks the bistable potential, we define the occupancy as the percent of oscillators occupying the lower well at the extremes of one forcing cycle (namely, at $\omega t = \pi/2$ and $\omega t = 3\pi/2$):

$$\text{occupancy} = (\%\text{ oscillators on left when left well is lower} + \%\text{ oscillators on right when right well is lower})/2\,. \quad (4)$$

At low noise the oscillators are confined to one well (say, the left) regardless of whether it is low or high and at occupancy $(100\% + 0\%)/2 = 50\%$. At high noise an oscillator is equally likely to be in either well so that occupancy is $(50\% + 50\%)/2 = 50\%$. At synchronization, the oscillators hop back and forth so as to remain always in the lower well and at occupancy $(100\% + 100\%)/2 = 100\%$.

Figure 4 illustrates how the occupancy of a chain (and hence its degree of synchronization) peaks at the same noise power as the SNR of a constituent oscillator. One can show that the occupancy is a measure of the power at the signal frequency, averaged over the array [8]. It essentially computes the power in one bin of the discrete Fourier transform, and hence is faster than the full Fourier transform. Since the occupancy is faster to calculate

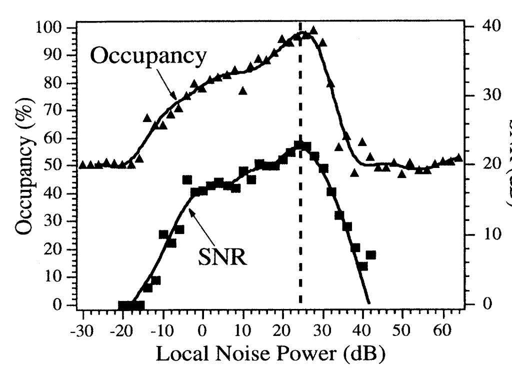

FIG. 4. Occupancy of a chain of 101 oscillators superimposed on the SNR of the middle oscillator of the chain. The (global) average occupancy of the lower well peaks at the same noise power as the (local) SNR of the middle (51st) oscillator. Coupling $\varepsilon = 1$. Other parameter values as in Fig. 1.

numerically than the SNR, it is a more convenient tuning parameter by which to optimize the response—both global (synchronization) and local (SNR)—of the chain.

In our numerical studies, the peak SNR of an oscillator coupled into an array corresponds (roughly) to a synchronous hopping rate of 2, namely once back and forth per forcing period. Indeed, the shift with coupling in the peak SNR mirrors the shift in the hopping rate: the larger the coupling, the more noise is required to achieve a given hopping rate. Furthermore, as soon as increasing noise forces the hopping rate past 2, the hopping rate diverges. This divergence naturally divides the dynamics into a potential-dominated regime (where the hopping rate remains bounded by 2) and a noise-dominated regime. The divergence in the hopping rate also reflects the relatively rapid decline in the occupancy function as the local noise power passes through its optimal value [10].

In summary, *spatiotemporal order and enhanced stochastic resonance can be induced in a locally and linearly coupled array of overdamped nonlinear oscillators by the optimization of a single adjustable parameter (noise or coupling).* Significant synchronization and enhancement can be obtained using even a small number of oscillators. Peak performance of a single oscillator corresponds to a global organization of the chain in space and time; noise, coupling, and bistable potential cooperate to create spatial order, temporal periodicity, and peak signal-to-noise ratio. We believe that these phenomena transcending our intentionally simple model are in fact quite general, and may be important in the design and operation of extended systems, from biological receptors to remote sensing arrays.

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