Model for Plane Turbulent Couette Flow

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Combining the Reynolds equations with the difference-quotient turbulence model a simple analytical theory of plane turbulent Couette flow has been obtained. In a very natural manner an order parameter of turbulence appears, which is related to the production rate of turbulent kinetic energy in the whole domain. Good agreement with experimental data has been achieved.

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Plane Couette flow occurs in a fluid between two plane parallel plates which are rectilinearly sheared. The flow can be obtained in two different ways, for example, by keeping one plate at rest and moving the second with a constant velocity U [Fig. 1(a)] and by movement of the plates in opposite directions with velocities $\pm U/2$ in reference to a laboratory system [Fig. 1(b)].

Laminar plane Couette flow yields the basis to explain Newton's friction law. This fundamental law of fluid dynamics is based on knowledge of a linear velocity profile occurring between adjacent layers as observed in flows dominated by viscous forces [Fig. 1(a)]. In turbulent states the mean flow profiles are of S shape [Fig. 1(b)]. Because plane Couette flow is very difficult to achieve experimentally relatively few experiments have been performed to study this flow type. For example, measurements have been reported by Reichhardt (see Refs. [1,2]), Robertson and Johnson [3], Aydin and Leutheusser [4], El Telbany and Reynolds [5], and recently by Bech *et al.* [6].

On the other hand, Couette flow probably is the most simple fundamental problem of turbulence, and therefore it is extremely valuable for a theoretical understanding of turbulent phenomena. Since von Kármán has pointed out its importance it has intrigued a number of researchers such as Kampé de Fériet, Heisenberg, Burger, and Mitchner, and Diessler and Fox (see Ref. [3]). These scientists were mainly motivated to work on Couette flow by the desire to understand homogenous turbulence. As we will show in the remainder of this Letter the production of turbulent kinetic energy—for finite Reynolds number flow—is nonzero and approximately constant over a large domain symmetrical to the center line between the two plane parallel plates. This leads to a homogenous turbulence field, which on the other hand—because of the mean shear flow—is nonisotropic. Therefore the results are quite different from those of plane turbulent Poiseuille flow, where the production even vanishes on the center line [7]. Nowadays by numerical simulations inherent structures of turbulence in Couette flows at modest and high Reynolds numbers can be studied more easily, e.g., large-scale structures, wall streaks, etc. (see Refs. [8,9]).

We begin our theoretical treatment by considering the Reynolds equations, which are presented in many textbooks on turbulence, e.g., in Ref. [10]. They are applied to the plane turbulent Couette flow problem showing a vanishing downstream pressure gradient. Furthermore, the dimensionless functions

$$\eta = \frac{x_2}{a}, \quad f_1(\eta, \mathrm{Re}^*) = \frac{\overline{u}_1}{u^*}, \quad f_{2,1}(\eta, \mathrm{Re}^*) = \frac{\overline{u_2' u_1'}}{(u^*)^2} \quad (1)$$

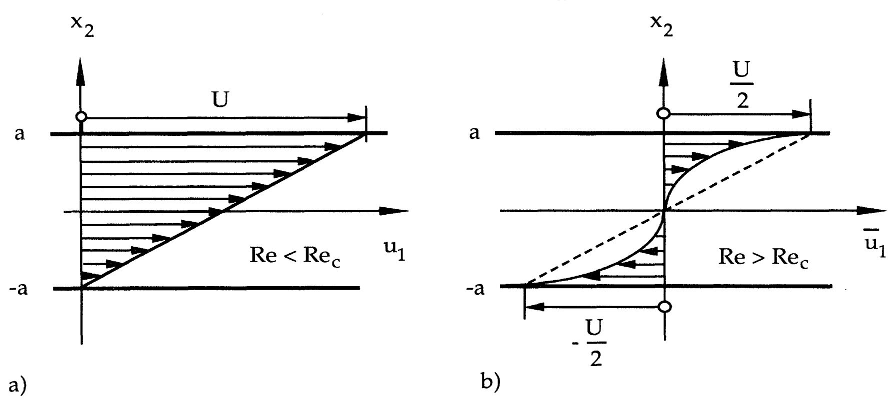

FIG. 1. Plane turbulent Couette flow between two rectilinearly sheared plates. The mean velocity profile of laminar flow (a) and turbulent flow (b) are shown in two different reference systems with a dimensionless mass flow equal to $\frac{1}{2}$ (a) and a vanishing mean mass flow in case (b).

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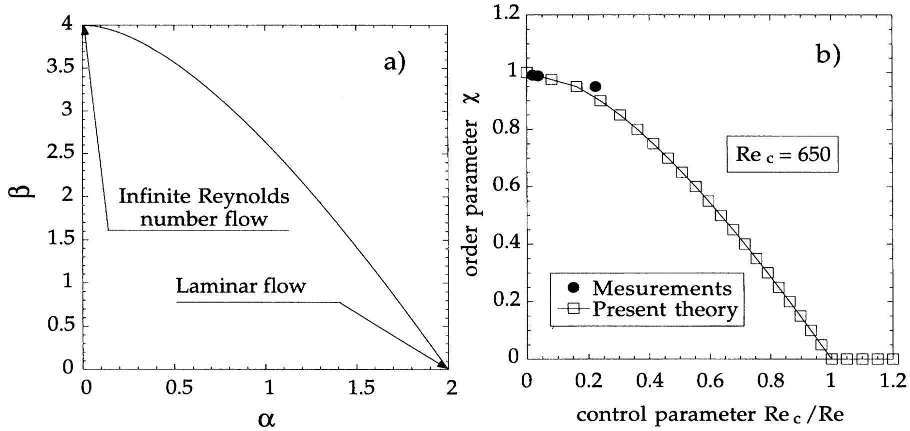

FIG. 2. Functional relation between the parameters α and β, which was calculated by Eq. (12) [see (a)]. On the right the order parameter occurring, $\chi = \beta/4$, is compared with experimental results from Ref. [2].

are taken into consideration. The quantity a denotes the half-width between the plane parallel plates. f_1 is the dimensionless velocity parallel to the inner surface of the plates and $f_{2,1}$ the Reynolds shear stress. In boundary-layer theories it is customary to define the characteristic velocity (τ_0 denotes the boundary shear stress)

$$u^* = \sqrt{|\tau_0|/\rho}, \quad \mathrm{Re}^* = au^*/\nu,$$
$$\mathrm{Re} = a\overline{u}_{1_{\max}}/\nu, \quad \overline{u}_{1_{\max}} = U, \tag{2}$$

which allows the definition of two different Reynolds numbers Re* and Re. After some calculations one obtains the following equation (see Ref. [11]):

$$\frac{1}{\mathrm{Re}^*}\frac{\partial f_1(\eta,\mathrm{Re}^*)}{\partial\eta} - f_{2,1}(\eta,\mathrm{Re}^*) - A(\mathrm{Re}^*)\eta - B(\mathrm{Re}^*) = 0. \tag{3}$$

The no-slip boundary condition is

$$f_1(-1,\mathrm{Re}^*) = 0, \quad f_1(+1,\mathrm{Re}^*) = \overline{u}_{1_{\max}}/u^*. \tag{4}$$

Furthermore, turbulent motion vanishes at the boundary by hindrance

$$f_{2,1}(-1,\mathrm{Re}^*) = f_{2,1}(+1,\mathrm{Re}^*) = 0. \tag{5}$$

By studying Eq. (3) at the boundaries one obtains $1 \mp A - B = 0$. After substituting the solutions of this equation $A = 0$ and $B = 1$ we obtain

$$\frac{1}{\mathrm{Re}^*}\frac{\partial f_1(\eta,\mathrm{Re}^*)}{\partial\eta} - f_{2,1}(\eta,\mathrm{Re}^*) = 1. \tag{6}$$

Eight years ago the difference-quotient turbulence model was developed (see Refs. [12,13]). It can be shown that this alternative closure corresponds to eddies with an infinite number of different scales. Furthermore, it relates to the turbulent kinetic energy of this infinite set. Compared with the Kolmogorov-Obukhov model (KOM) [14] it contains some interesting generalizations which are essential for the description of nonisotropic turbulent shear flows. The length scales are functions of space coordinates whereas in the KOM length scales vary only from one generation of eddies to the next but show no further dependence on the position in the flow. Furthermore, the difference-quotient turbulence model

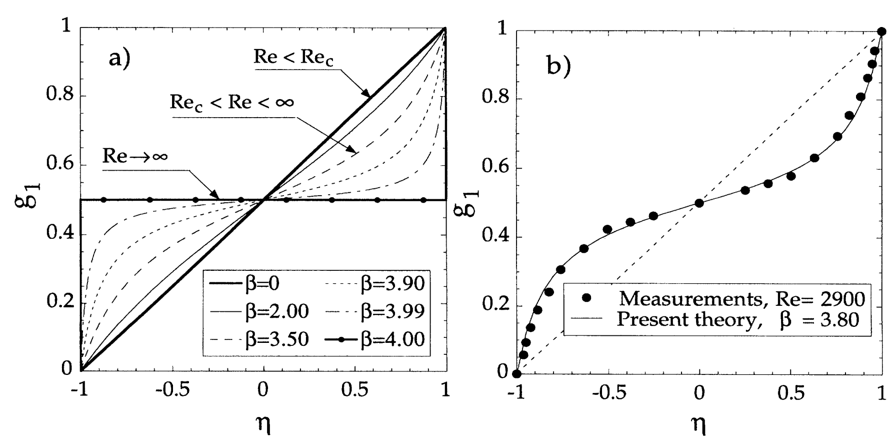

FIG. 3. Time-averaged velocity profiles for different turbulence intensities. The limiting cases of laminar flow and infinite Reynolds number flow are presented by thicker solid lines (a). In the second figure (b) as an example a comparison with experimental results from Ref. [2] is shown.

not only refers to the turbulent kinetic energy contained in all the "scales" (Taylor's energy spectrum), but it additionally is composed of interaction energies between eddies of different "diameters." Probably the most compact mathematical description of the proposed closure is the following difference-quotient representation:

$$\overline{u_2'u_1'}(x_1,x_2) = -\sigma\chi_2[\overline{u}_1(x_1,x_2) - \overline{u}_{1_{\min}}(x_1)] \times \frac{\overline{u}_{1_{\max}}(x_1) - \overline{u}_1(x_1,x_2)}{x_{2_{\max}} - x_2}. \quad (7)$$

The characteristic length of the flow problem χ_2 perpendicular to the main flow direction is $a - x_2$. The "mixing length" is chosen to be equivalent to the distance from the driving wall. The quantity $x_{2_{\max}}$ denotes the space coordinate, where $\overline{u}_1$ takes its maximum or supremum as a function of the variable x_2 and therefore is equal to a. Then we simply obtain

$$\overline{u_2'u_1'}(x_1,x_2) = -\sigma[\overline{u}_1(x_1,x_2) - \overline{u}_1(x_1,-1)] \times [\overline{u}_1(x_1,+1) - \overline{u}_1(x_1,x_2)], \quad (8)$$

which fulfills all the desired invariance and symmetry requirements. With the following definitions:

$$g_1(\eta) = f_1(\eta)/f_1(1), \quad g_{2,1}(\eta) = f_{2,1}(\eta), \quad \alpha = f_1(1)/\mathrm{Re}^*, \quad \beta = \sigma f_1(1)^2, \quad (9)$$

a differential equation and two boundary conditions are obtained

$$\alpha g_1'(\eta) + \beta g_1(\eta) - \beta g_1(\eta)^2 - 1 = 0, \quad g_1(-1) = 0, \quad g_1(1) = 1. \quad (10)$$

The result for the mean flow profile can be obtained by elementary techniques

$$g_1(\eta) = \frac{1}{2}\left\{1 + \sqrt{\frac{4-\beta}{\beta}}\tan\left[\arctan\left(\sqrt{\frac{4-\beta}{\beta}}\right)\eta\right]\right\}. \quad (11)$$

To obtain Eq. (11) the following function, which follows from the requirements (10), was substituted into the general solution of problem (10):

$$\alpha = \left[\sqrt{(4-\beta)/\beta} \Big/ 2\arctan\sqrt{(4-\beta)/\beta}\right]\beta. \quad (12)$$

The inverse function $\beta(\alpha)$ is shown in Fig. 2(a). After some calculations one can show that

$$\mathrm{Re}^* = \frac{\arctan\sqrt{\chi/(1-\chi)}}{\sqrt{1-\chi}}\frac{1}{\sqrt{\sigma}}, \quad (13a)$$

$$\frac{\mathrm{Re}_c}{\mathrm{Re}} = \frac{\sqrt{\chi(1-\chi)}}{\arctan\sqrt{\chi/(1-\chi)}}, \quad \chi = \frac{\beta}{4}, \quad (13b)$$

where an order parameter χ of turbulence has been introduced. Equation (13b) is shown in Fig. 2(b).

The mean flow profiles calculated with Eq. (11) are shown in Fig. 3(a) and a comparison with experimental results can be seen in Fig. 3(b). The point-wise limits of the flow profiles for $|\eta| \leq 1$ are

$$\beta \to 0 \Rightarrow \alpha \to 2, \quad g_1 \to \frac{1}{2}(1+\eta) \text{ point-wise for } |\eta| \leq 1, \quad (14a)$$

$$\beta \to 4 \Rightarrow \alpha \to 0, \quad g_1 \to \frac{1}{2}[H(1-\eta) + H(\eta+1)] \text{ point-wise for } |\eta| < 1, \quad (14b)$$

where H describes a Heaviside distribution in S. Therefore the derivative of the mean flow profile consists of two delta distributions at the positions of the two shearing plates. Substituting Eq. (11) into the dimensionless form of Eq. (8) one obtains

$$g_{2,1} = -\beta g_1(1-g_1) = -\frac{\beta}{4}\left\{1 - \left(\frac{4-\beta}{\beta}\right)\tan^2 \times \left[\arctan\left(\sqrt{\frac{\beta}{4-\beta}}\right)\eta\right]\right\}, \quad (15)$$

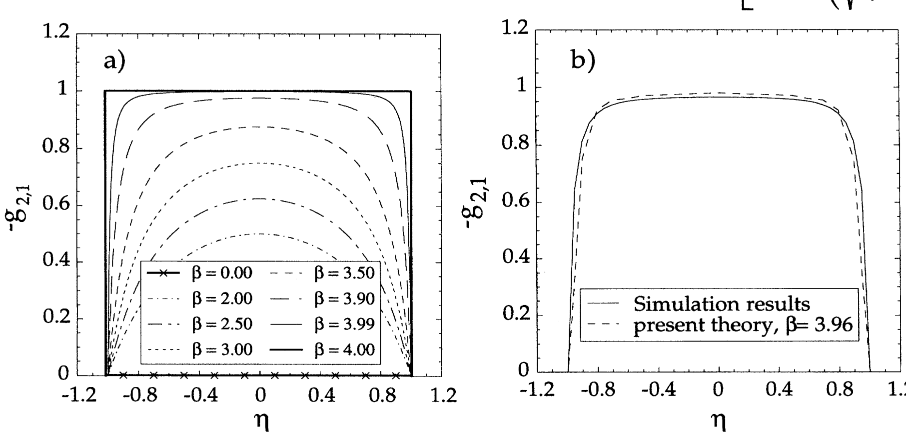

FIG. 4. Reynolds shear stress for different order parameters β. Limiting cases of laminar and infinite Reynolds number flow are shown by thicker solid lines (a) and in (b) a comparison of analytical calculations with latest simulations results [15] are shown.

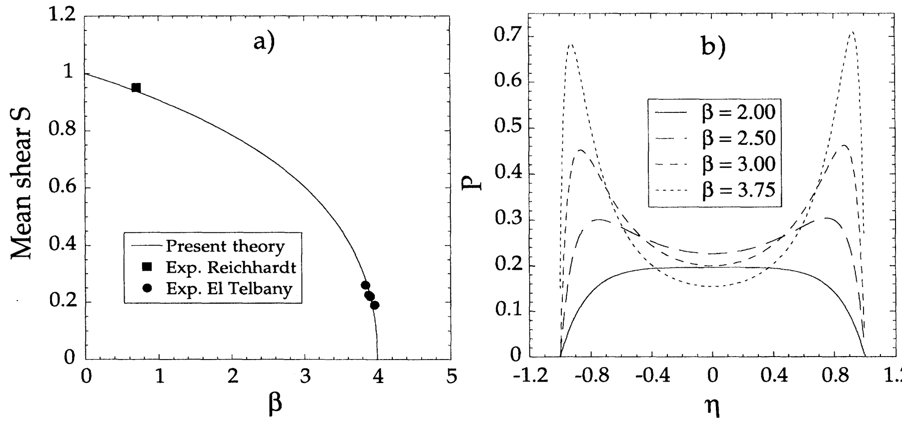

FIG. 5. In (a) the calculated mean shear S is compared with experimental results from Refs. [2,5]. The production rate of turbulent kinetic energy P is shown in (b) on the right. In some cases this quantity is approximately constant in broad domains symmetrical to the center line. Note that when β tends towards 4, the production of turbulent kinetic energy P converges towards two delta distributions at the positions of the bounding plates.

the Reynolds shear stress which is shown in Fig. 4(a). Note that in Fig. 4(b) $\beta = 3.96$ has been determined with Eq. (13b). In addition, with this value very good agreement between the measured and calculated mean velocity profiles has been obtained. Therefore the comparison shown in Fig. 4(b) is a very reliable test of the performance of the turbulence model under consideration.

The mean shear S is defined by

$$S = 2\frac{dg_1}{d\eta}\Big|_{\eta=0} = \sqrt{\frac{4-\beta}{\beta}}\arctan\left(\sqrt{\frac{\beta}{4-\beta}}\right). \quad (16)$$

This is shown in Fig. 5(a). The production of the turbulent kinetic energy, referring to Ref. [10], is [see also Fig. 5(b)]

$$P = -g_{2,1}\, dg_1/d\eta\,, \quad (17)$$

which can easily be calculated by combining the derivative of Eq. (11) with (15b). From Eqs. (11), (15a), and (17), however, we conclude directly that

$$\int_{-1}^{+1} P\, d\eta = \beta\int_0^{+1}(g_1 - g_1^2)\, dg_1 = \frac{\beta}{6}. \quad (18)$$

The parameter β is proportional to the production of turbulent kinetic energy in the entire domain per unit length in the downstream direction. Therefore it is a very meaningful property, with all the desired features of an order parameter (characterizing turbulence). This is not very surprising as turbulent fields are composed of two "phases," namely, laminar patches separated from turbulent regions by fractal boundaries. With increasing Reynolds number the laminar regions become smaller and less until they completely vanish, and in the infinite Reynolds number limit—in a statistical sense—again a higher degree of symmetry is obtained [16].

The results of Poiseuille and Couette flow show striking analogies to the theory of critical phenomena. Most experiments on Couette flow which have been reported in the past are related to Reynolds numbers between 3000 and 35 000. For this domain the order parameter varies in the range 3.8 to 4.0. Therefore our results give evidence that—at least from a disorder or order point of view—many more experiments should be performed with Reynolds numbers only slightly above criticality ($650 \leq \mathrm{Re} \leq 3000$). Then the presented theory could be tested much more thoroughly.

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