Controlling Chaos using Differential Geometric Method

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We present an effective approach for controlling chaos by using a differential geometric method. It has been shown that the proposed method can control chaotic motion not only to a steady state but also to any desired periodic orbit. The main characteristic of the method is to algebraically transform a nonlinear dynamics into a linear one, so that linear control techniques can be applied. To demonstrate the feasibility of our proposed method, a Lorenz system under different designed requirements is illustrated.

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Chaotic behavior occurs in many mechanical or electric oscillators, in rotating or heated fluids, in chemical reactions, in laser cavities, etc. However, these irregular and complex phenomena are often undesirable. In many practical situations, in order to improve system performance or avoid fatigue failures of mechanical systems, we must control a chaotic system to a periodic orbit or a steady state, but particularly the last one. Therefore how to control chaotic systems has received increased attention [1–12]. The pioneers Ott, Grebogi, and Yorke (OGY) proposed an efficient method to achieve this control in which only a small time-dependent perturbation is made on one of the accessible system parameters [1]. Modified methods and other approaches are presented continuously [2–12]. The OGY method can convert chaotic motion to one of a large number of unstable periodic orbits embedded within a strange attractor. With delay coordinate embedding, the OGY method is applicable to experimental situations in which *a priori* analytical knowledge of the system dynamics is not available. Nevertheless, the application of the OGY method is limited due to the measurement errors. In the chaotic state of a nonlinear dynamic system this error is amplified exponentially with time such that the trajectory may not be predicted with the precision necessary for the OGY method. For this reason Hübinger, Doerner, and Martienssen [5] proposed an alternative method which allows a "nearly" continuous "local" control by adjusting of the control parameter concurrently. However, for some systems which are perturbed by the environment or other factors, the above two methods fail.

In addition to the above methods, some researchers [6,7] applied a periodic external force to eliminate chaos in a dynamic system. Similarly, those approaches can change a strange attractor to a periodic orbit but not a steady state. To solve this problem, the occasional feedback technique or conventional linear feedback methods are feasible approaches [8–12]. But the closed-loop system is still nonlinear and the feedback gains must be trial and error. Otherwise, dynamic analysis of the closed-loop system is necessary; this is time consuming and may be impractical due to the difficulty of obtaining an exact physical model.

Another possible method to control chaos is feedback linearization [13,14]. The idea of the approach is to algebraically transform a nonlinear system dynamics into a linear one, so that linear control techniques can be applied. This differs from conventional linearization (i.e., Jacobian linearization) in that feedback linearization is achieved by exact state transformations and feedback, rather than by linear approximations of the dynamics. The ability to use feedback to transform a nonlinear system into a controllable linear system by canceling nonlinearities requires the nonlinear system to have the structure

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\beta^{-1}(\boldsymbol{x})[u - \alpha(\boldsymbol{x})]. \tag{1}$$

If the nonlinear state equation does not have the structure of (1), this does not mean that we cannot linearize the system via feedback. In this paper we present the basic principles of a differential geometric method and their applications in controlling chaos. This method can relax the match condition that nonlinear systems must have the structure of equation (1) [15]. By using this approach, we can control a chaotic system to any desired steady state or arbitrary periodic orbit; furthermore, the transient response of the controlled system can be designed to satisfy the performance requirement.

First, let S_1 and S_2 be two systems defined, respectively, by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u) \tag{2}$$

and

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}, \nu). \tag{3}$$

We say that S_1 is related to S_2 if there is a C^∞ diffeomorphism $T\colon \mathrm{R}^{n+1} \to \mathrm{R}^{n+1}$ with $T_1(\boldsymbol{x}, u) = y_1, \dots, T_n(\boldsymbol{x}, u) = y_n$, and $T_{n+1}(\boldsymbol{x}, u) = \nu$ such that for every state-control trajectory $(\boldsymbol{x}(t), u(t))$

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of S_1 we have the corresponding $T_1(t), \dots, T_n(t), T_{n+1}$ satisfying Eq. (3), namely,

$$\dot{T}_1 = g_1(T_1, \dots, T_n, T_{n+1}),$$
$$\vdots$$
$$\dot{T}_n = g_n(T_1, \dots, T_n, T_{n+1}).$$

In other words, $(T_1(t), \dots, T_n(t), T_{n+1}(t))$ is a state-control trajectory of S_2. The set of such transformations is denoted by $\mathcal{T}$, and it also can be said that S_1 is $\mathcal{T}$ equivalent to S_2. Our interest is in the $\mathcal{T}$-equivalence class of systems which contains controllable linear systems. Su [15] showed that every controllable linear system is $\mathcal{T}$ equivalent to the systems

$$\begin{bmatrix} \dot{y} \\ \vdots \\ \dot{y}_{n-1} \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} y_2 \\ \vdots \\ y_n \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \nu . \tag{4}$$

There is a transformation T for the given system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\phi(\boldsymbol{x}, u)$$

in S_0 such that

$$\langle dT_i, \boldsymbol{f} + \boldsymbol{g}\phi \rangle = T_{i+1}, \quad i = 1, \dots, n,$$

where dT_i denotes the gradient of T_i and $\langle dT_i, \boldsymbol{f} \rangle$ is a scalar field defined by

$$\langle dT_i, \boldsymbol{f} \rangle = \frac{\partial T_i}{\partial x_1} f_1 + \cdots + \frac{\partial T_i}{\partial x_n} f_n .$$

Since $T_1, \dots, T_n$ are independent of u, we conclude that

$$\langle dT_i, \boldsymbol{g} \rangle = 0,$$
$$\langle dT_i, \boldsymbol{f} \rangle = T_{i+1}, \quad i = 1, \dots, n - 1,$$

and

$$\langle dT_n, \boldsymbol{f} + \boldsymbol{g}\phi \rangle = \langle dT_n, \boldsymbol{f} \rangle + \langle dT_n, \boldsymbol{g} \rangle \phi = T_{n+1} = \nu .$$

Up to the present, many efficient identification methods for nonlinear systems [18,19] have been proposed that support the feasibility of our method. Moreover, in practical applications, our proposed method allows parametric variations and the existence of high frequency unmodeled structures. In order to show the feasibility of our proposed method in this paper, let us consider a Lorenz system [16,17] which can be described by

$$\dot{x} = -px + py,$$
$$\dot{y} = -xz - y,$$
$$\dot{z} = xy - z - R,$$

where $R = R_0 + u$ is the Rayleigh number, R_0 is our operation value, $p = 10$ is the Prandtl number, and u is the control parameter. The physical meaning of the variables (x, y, z) can be seen in Ref. [16]. If $R_0 = 28$, the uncontrolled system (i.e., $u = 0$) is chaotic and there are three unstable equilibrium points $(C_0, C_0, -1)$, $(0, 0, -R_0)$, and $(-C_0, -C_0, -1)$, where $C_0 \equiv \sqrt{R_0 - 1}$. If the state $(x, y, z) = (C_0, C_0, -1)$ is our set point, from the state space it is easy to observe that the strange attractor of the Lorenz system does not include the desired equilibrium point. Also, it is undefined in the Poincaré map for this point. Therefore the OGY method is not applicable in this case.

We now use the differential geometric method to control the Lorenz system. First, let us define a new set of dependent variables $(x_1, x_2, x_3) = (x - C_0, y - C_0, z + 1)$; then the system can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -px_1 + px_2 \\ x_1 - x_2 - C_0x_3 - x_1x_3 \\ C_0(x_1 + x_2) - x_3 + x_1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u, \tag{5}$$

and the new uncontrolled system has three unstable equilibrium points (0,0,0), $(-C_0, -C_0, 1 - R_0)$, and $(-2C_0, -2C_0, 0)$, where the point $(x_1, x_2, x_3) = (0, 0, 0)$ is the target state. According to the previous descriptions, it can be verified that system (5) is $\mathcal{T}$ equivalent to system (4). The desired transformation can be obtained as

$$T_1 = x_1,$$
$$T_2 = -px_1 + px_2,$$
$$T_3 = (p^2 + p)(x_1 - x_2) - pC_0x_3 - px_1x_3,$$

and

$$T_4 = \nu = \langle dT_3, \boldsymbol{f} \rangle + \langle dT_3, \boldsymbol{g} \rangle u,$$

where

$$\begin{aligned} \langle dT_3, \boldsymbol{f} \rangle \equiv q(\boldsymbol{x}) = {} & (p^2 + p - px_3)(-px_1 + px_2) \\ & - (p^2 + p)(x_1 - x_2 - C_0x_3 - x_1x_3)x_2 \\ & - (pC_0 + px_1)(C_0x_1 + C_0x_2 - x_3 + x_1x_2), \\ \langle dT_3, \boldsymbol{g} \rangle \equiv {} & s(\boldsymbol{x}) = p\,(x_1 + C_0). \end{aligned}$$

If we apply the linear feedback control $\nu = -\boldsymbol{K}\boldsymbol{y} = -1000y_1 - 215y_2 - 17.5y_3$ such that all eigenvalues of the system have negative real parts, then $\boldsymbol{y} = \boldsymbol{0}$ (or equivalently $\boldsymbol{x} = \boldsymbol{0}$) is an asymptotically stable equilibrium point of the linear system. The schematic diagram of the feedback control system is shown in Fig. 1. Since

$$u = -\frac{q(\boldsymbol{x})}{s(\boldsymbol{x})} + \frac{\nu}{s(\boldsymbol{x})},$$

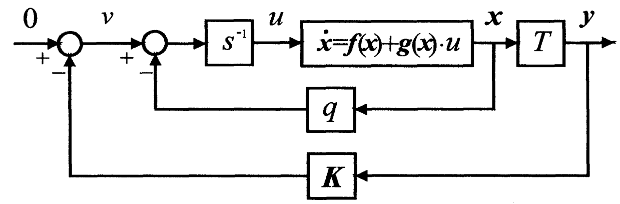

FIG. 1. Schematic description of the feedback control system.

to avoid $s(\boldsymbol{x})$ being singular and to limit the control energy, let the controller begin to work when $s(\boldsymbol{x}) > 0$ and the range in which we are allowed to vary u be $-R_m < u < R_m \leq R_0$, where R_m is an appropriate positive number. Notice that, in the previous descriptions, it is assumed that everything on a Euclidean space is defined in the neighborhood of the origin. But, in practical situations, such a linearized method usually holds not only in the neighborhood of the origin. For example, in the Lorenz system described above, the linearization is valid when $s(\boldsymbol{x}) > 0$ (or $x_1 > -C_0$). Thus any initial state within the range can be controlled to another one of the same range under energy limitation.

In the following, several numerical examples under different situations are illustrated. First, we take $R_m = R_0$. The simulated result is shown in Fig. 2. [In Figs. 2–5, the controller has been switched on at a nondimensional time $t = 25$. The upper trace is the response of R; the lower one is the system output $x(t)$.] Since the closed-loop system is linear, the system response is optimal due to the feedback control. As was expected, the control signal u is near zero (or equivalently $R = R_0$) when the transient response of the system comes into the desired steady state. We stress that larger control energy is inevitable in the beginning (no matter what controller we use); this is because the initial state is far from the desired one. Because the large control energy is allowable, our method can immediately converge a large error to zero instead of waiting for the actual state to reach the neighborhood of the desired state, and the robustness is enhanced. Next, we consider the effect of measurement noise. We add the independent noise $\varepsilon\delta_1$, $\varepsilon\delta_2$, and $\varepsilon\delta_3$ to the measurements x_1, x_2, and x_3, respectively. The random noises δ_1, δ_2, and δ_3 are normally distributed, and have mean value 0 and variance 0.5. Figure 3 shows

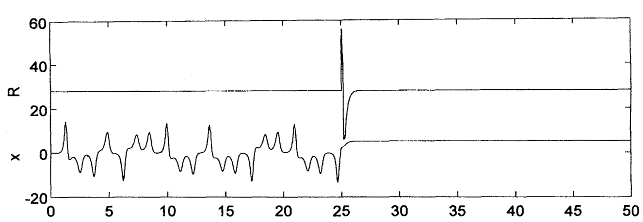

FIG. 2. Time responses of the Lorenz system without noise.

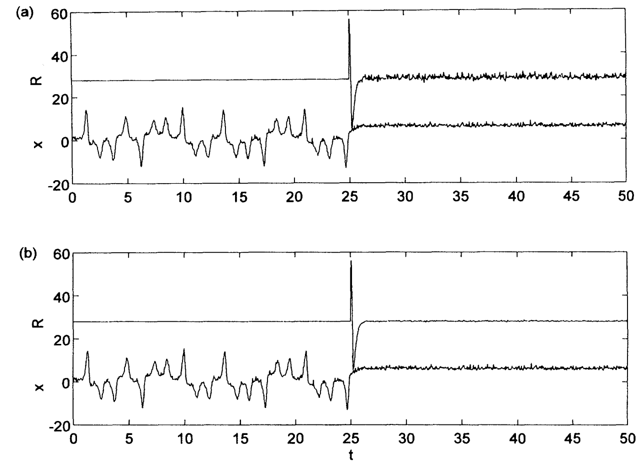

FIG. 3. Time responses of the Lorenz system with two different levels of noise. (a) $\varepsilon = 0.05$. (b) $\varepsilon = 0.01$.

the results of the stabilization of the unstable equilibrium point $(x, y, z) = (C_0, C_0, -1)$ of the Lorenz system for two different levels of noise. The increase in noise leads to the increase of the amplitude of perturbation but not to occasional bursts of the system into the region far from the desired state. If we further reduce the allowable varied range of u, i.e., decrease the value of R_m to 5.0, the result is given in Fig. 4 before controlling and after controlling. Because of the limitation of control energy, the system response has transient oscillation after controlling, but the steady response keeps zero error all the same. In practical applications, the system may be perturbed by parametric uncertainty or process noise. To consider this situation, besides the measurement noise, we let the Prandtl number vary randomly in the interval [8.0,12.0] and add terms $\varepsilon'\delta_1$, $\varepsilon'\delta_2$, and $\varepsilon'\delta_3$ to the right-hand sides of Eq. (6), in which we let $\varepsilon' = 4.0$. The controller is designed according to the nominal system ($p = 10$). The result of the numerical simulation is depicted in Fig. 5. It shows that the steady response is hardly affected by the perturbations. Notice, that for such a noisy situation the OGY method and many other approaches are not feasible.

In addition to stabilizing an unstable equilibrium point, if we let the reference input $r(t)$ be a desired periodic

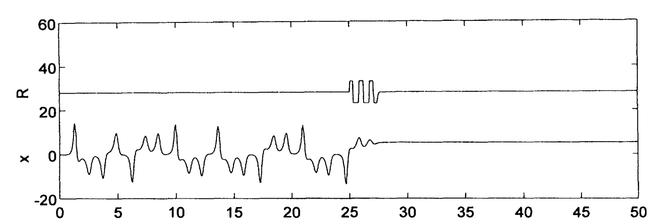

FIG. 4. Time responses of the Lorenz system where the allowable varied range of u is decreased (i.e., the value of R_m is reduced to 5.0).

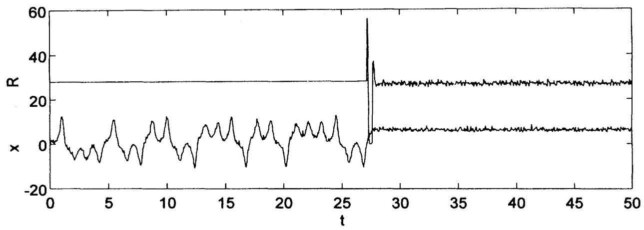

FIG. 5. Time response of the Lorenz system with parametric uncertainty and process noise.

signal, the system output will track the reference orbit. The results of tracking different periodic orbits are shown in Fig. 6.

In conclusion, an approach by using a differential geometric method to control chaos has been presented. It has been shown that the proposed method can control chaotic motion not only to an equilibrium point but also to any desired periodic orbit. A Lorenz system demonstrating the feasibility of our proposed method has been illustrated. Some advantages of this method are the following. (1) The control strategy of our proposed method is easier to implement; (2) it can perform jobs automatically after being designed and implemented and can stabilize the overall control system efficiently; (3)

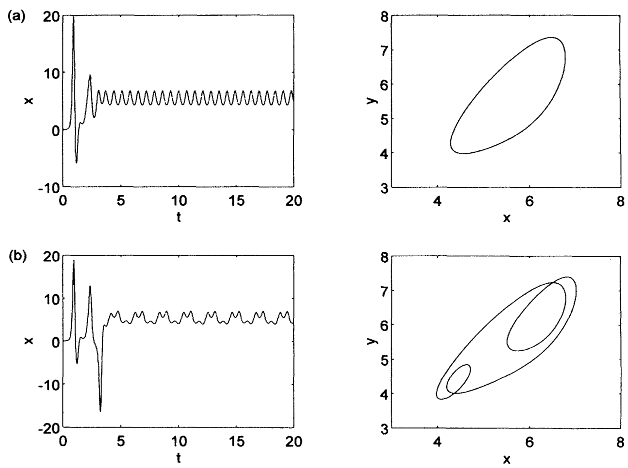

FIG. 6. Results of tracking different periodic orbits. (a) Period 1. (b) Period 3.

only nominal systems have to be known (identified) in advance (in practice, allowing the existence of parameter uncertainty and high-frequency unmodeled dynamics); (4) the reference input can be any steady state or arbitrary periodic orbit even outside the strange attractor; (5) the dynamic characteristics of controlled systems are linear and their transient responses can be set according to the requirements of designers; (6) the control force is very small (i.e., the required parameter perturbation is small) when the system response is close to the designed trajectory; (7) control can be achieved even with large noise or bounded control energy; (8) the converging speed of error is very fast; and (9) the larger the allowable control energy, the more robust the controlled system.

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