Self-Similarity and Localization

Jukka A. Ketoja
Department of Physics, Åbo Akademi, Porthansgatan 3, FIN-20500 Åbo, Finland

Indubala I. Satija*
Department of Physics and Institute of Computational Sciences and Informatics, George Mason University, Fairfax, Virginia 22030

(Received 25 April 1995)

The localized eigenstates of the Harper equation exhibit universal self-similar fluctuations once the exponentially decaying part of the wave function is factorized out. For a fixed quantum state, we show that the whole localized phase is characterized by a single strong coupling fixed point of the renormalization equations. This fixed point also describes the generalized Harper model with next nearest neighbor interaction below a certain threshold. Above the threshold, the fluctuations in the generalized Harper model are described by a strange invariant set of the renormalization equations.

PACS numbers: 75.30.Kz, 64.60.Ak, 64.60.Fr

In the extensively studied Harper equation [1]

$$\psi_{i+1} + \psi_{i-1} + 2\lambda \cos[2\pi(i\sigma + \phi)]\psi_i = E\psi_i\,, \quad (1)$$

with σ equal to the inverse golden mean, most of the attention has been focused on the onset of metal-insulator transition at $\lambda_c = 1$, where the quantum states and the spectrum exhibit self-similarity and are multifractal [2]. In this paper, we show that the fluctuations in the localized wave functions of the model for $\lambda > 1$ possess the same complexity and richness as the critical states. The universal self-similar fluctuations at the band edges are determined by the strong coupling fixed point of a renormalization operator. We solve for this nontrivial fixed point and obtain the universal scaling ratio for the fluctuations. In addition, the stability of the fixed point is analyzed by linearizing the renormalization transformation. In particular, we study the perturbation associated with a generalized Harper equation describing a two-dimensional electron gas with next nearest neighbor (NNN) interaction in the presence of an inverse golden mean magnetic flux [see Eq. (12)] [3,4]. The universality class for the fluctuations turns out to be unaltered as long as the NNN coupling is below a certain threshold. Above the threshold, the situation is a lot more complicated as the renormalization flow seems to converge on an invariant strange set.

The wave function ψ_i is written as

$$\psi_i = e^{-\gamma|i|}\eta_i\,, \quad (2)$$

where γ is the Lyapunov exponent which vanishes in the extended (E) and the critical (C) phase. The localized (L) phase is characterized by a positive Lyapunov exponent corresponding to the exponential decay of the wave function. It is assumed that the phase ϕ in Eq. (1) is chosen so that the main peak of the wave function is at $i = 0$ so that η_i is bounded [5,6]. For the Harper equation it has been shown analytically that $\gamma = \ln(\lambda)$ in the L phase [3,7]. Therefore, the function η_i describing the fluctuations in the exponentially decaying wave function satisfies the following tight binding model (TBM) for $i > 0$ [8]:

$$\frac{1}{\lambda}\eta_{i+1} + \lambda\eta_{i-1} + 2\lambda\cos[2\pi(i\sigma + \phi)]\eta_i = E\eta_i\,. \quad (3)$$

We study this TBM using our recently developed decimation approach [5,9], where all sites except those labeled by the Fibonacci numbers F_n are decimated. At the nth decimation level, the TBM is expressed in the form [10]

$$f_n(i)\eta(i + F_{n+1}) = \eta(i + F_n) + e_n(i)\eta(i)\,. \quad (4)$$

The additive property of the Fibonacci numbers provides exact recursion relations for the decimation functions e_n and f_n:

$$e_{n+1}(i) = -\frac{Ae_n(i)}{1 + Af_n(i)}\,, \quad (5)$$

$$f_{n+1}(i) = \frac{f_{n-1}(i + F_n)f_n(i + F_n)}{1 + Af_n(i)}\,, \quad (6)$$

$$A = e_{n-1}(i + F_n) + f_{n-1}(i + F_n)e_n(i + F_n)\,.$$

In this approach, the critical phase with self-similar wave functions is characterized by a nontrivial asymptotic p cycle (with the length p equal to 3 or 6 for the Harper model) for the decimation functions [5]. On the other hand, the asymptotic behavior of $e_n(0)$ and $f_n(0)$ determines the universal scaling ratios

$$\zeta_j = \lim_{n\to\infty} |\eta(F_{pn+j})/\eta(0)|\,, \quad j = 0, \ldots, p - 1\,. \quad (7)$$

We apply the decimation method to study the fluctuations of the L phase where Eq. (3) provides the initial conditions for the decimation functions $e_2(i)$ and $f_2(i)$ in

 0031-9007/95/75(14)/2762(4)$06.00

addition to the trivial conditions $e_1 \equiv 0$ and $f_1 \equiv 1$. As $f_2 \sim 1/\lambda < 1$ in the L phase, the recursion relations suggest that $f_n \sim 1/\lambda^{F_{n-1}}$, i.e., $f_n \rightarrow 0$ as $n \rightarrow \infty$. This was confirmed by the numerical iteration of Eqs. (5) and (6). Because f_n vanishes asymptotically, the scaling ratio ζ_j can be obtained directly as the limit of $|e_{pn+j}(0)|$ as n tends to infinity. At the band edges, we find that $|e_n(0)|$ converges to the fixed point 0.1726 for all $\lambda > 1$. Therefore, the fluctuations in the localized wave functions are universal throughout the L phase and are determined by the strong coupling fixed point of the system (see Fig. 1).

Taking the limit $\lambda \rightarrow \infty$, the TBM reduces to

$$\eta_{i-1} + v\cos[2\pi(i\sigma + \phi)]\eta_i = \epsilon\eta_i\,, \tag{8}$$

with $v = 2$ and $\epsilon = \lim_{\lambda\rightarrow\infty} E/\lambda$. For the lower (upper) band edge, ϵ is equal to -2 (2). At the band center, $\epsilon = 0$ and Eq. (8) is identical to the quantum Ising model in a quasiperiodic transverse field at the onset of long range order [5]. Therefore, the fluctuations at the band center in the L phase are described by the conformal universality class of the Ising model [5]. $|e_n(0)|$, and thus the scaling factor ζ, exhibits asymptotically a universal 3-cycle 0.2307, 0.5904, and 0.2683.

In order to solve for the strong coupling fixed point at the band edges (the 3-cycle at the band center could be analyzed in a similar way), we first notice that Eq. (5) can be simplified into

$$e_{n+1}(i) = -e_{n-1}(i + F_n)e_n(i)\,, \tag{9}$$

assuming that $f_n \equiv f_{n-1} \equiv 0$. Therefore this simple recursion relation describes both the asymptotic behavior

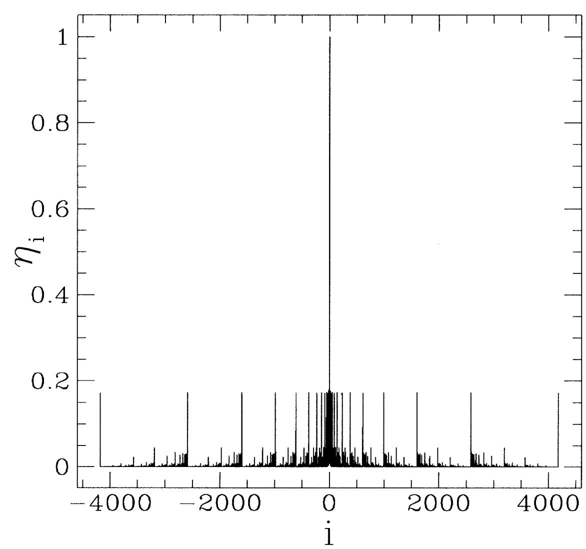

FIG. 1. The absolute value of the universal fluctuations of a band-edge eigenstate in the L phase of the Harper equation. Note that unlike the critical phase wave function, the L phase fluctuations are not symmetric about the secondary peaks.

as $n \rightarrow \infty$ for any $\lambda > 1$ and the limit $\lambda \rightarrow \infty$, where $f_n \equiv 0$ for all $n > 1$. By replacing the discrete lattice index i by the continuous renormalized variable $x = (-\sigma)^{-n}\langle i\sigma\rangle$, where $\langle\ \rangle$ denotes the fractional part [5], Eq. (9) transforms into the form

$$e_{n+1}(x) = -e_{n-1}(\sigma^2 x + \sigma)e_n(-\sigma x)\,. \tag{10}$$

The high n limit can be studied by introducing the renormalization operator

$$T[u(x), t(x)] = [t(-\sigma x), -u(-\sigma x - 1)t(-\sigma x)]\,. \tag{11}$$

The importance of this operator follows from the fact that the recursion (10) can be represented as the operation by T on the pair $[e_{n-1}(-\sigma x), e_n(x)]$. Moreover, the limiting function $e^*(x) = \lim_{n\rightarrow\infty} e_n(x)$ corresponds to the fixed point $[e^*(-\sigma x), e^*(x)]$ of the operator T. An initial estimate of the fixed point can be obtained by applying T subsequently on the pair $[e_2(-\sigma x), e_3(x)]$ obtained from Eq. (8). It turns out that it is easier to expand $1/e^*(x)$ than $e^*(x)$ (they both satisfy the same fixed point equation), so we solve for the coefficients of $1/e^*(x)$ by truncating the series and applying the Newton method to determine the fixed point of T. The power series is convergent in the domain $|x| \leq 1$, and we can obtain better and better estimates for the principal scaling ratio $\zeta = |e^*(0)|$ by increasing the order of the power series. Including terms up to the order x^{23}, we observe that $e^*(0)$ approaches $-0.172\,586\,410\,945 \pm 10^{-12}$ in agreement with the result obtained by iterating the decimation equations [11].

The linear stability analysis at the fixed point shows that the renormalization operator T has the unstable eigenvalues $\pm\sigma^{-2}$ and $\pm\sigma^{-1}$ (σ^{-1} is a double eigenvalue) and the marginal eigenvalue -1. In addition, there is a set of stable eigenvalues which are powers of the inverse golden mean. It should be noted that our renormalization operator resembles the one of Ostlund and Pandit [6] for the study of the critical point of the Harper equation. Although in their case t and u are 2×2 matrices, the eigenvalue analysis is similar in their and our cases. The variation of ϵ or v in Eq. (8) leads to an asymptotic escape from the fixed point in the eigendirection associated with the unstable eigenvalue σ^{-2}. In the same way, the variation of ϕ can be related to the eigenvalue $-\sigma^{-1}$. We expect that some of the remaining unstable eigenvalues and the marginal one are not "physical" because they represent variations which are inaccessible by using a TBM to define the pair $[u, t]$ (see Ref. [6]). We could further generalize the analysis by considering the direction of a finite decimation function f_n in function space, but as explained previously, the perturbation in that direction is expected to be irrelevant (with eigenvalue 0).

It is a characteristic feature of the strong coupling fixed point that all eigenvalues are powers of the golden mean. In the critical case, where the spectrum is singular continuous, there is a nontrivial eigenvalue associated with change of energy. We do not expect such an eigenvalue to appear in the localized phase where the spectrum is pointlike. In fact, the appearance of the eigenvalue σ^{-2} can be traced to the dual system [7] for which $\lambda \rightarrow \infty$ is the weak coupling limit [6].

We next study the generalized Harper equation

$$\begin{aligned}\{1 + \alpha\cos[2\pi(\sigma(i + \tfrac{1}{2}) + \phi)]\}\psi_{i+1} \\ +\{1 + \alpha\cos[2\pi(\sigma(i - \tfrac{1}{2}) + \phi)]\}\psi_{i-1} \\ +2\lambda\cos[2\pi(\sigma i + \phi)]\psi_i = E\psi_i \qquad (12)\end{aligned}$$

describing Bloch electrons on a square lattice with nearest neighbor (NN) coupling anisotropy λ and NNN coupling α. This model was recently studied using analytical and numerical methods [3] as well as applying the decimation scheme [4]. The model was found to exhibit a fat C phase provided $\alpha \geq 1$ and $\lambda \leq \alpha$. In the fat C phase the model exhibited various new universality classes: At certain specific values of λ and α, the self-similarity of the critical wave functions was described by higher order unstable (with respect to a change of parameter values) limit cycles of the renormalization group (RG) equations, while for arbitrary values of the parameters, the RG flow converged on an invariant set. This implied that the fractal characteristics of the wave functions were not self-similar. Since the cycle lengths were as high as 24, the question of which parameter values exhibited a limit cycle and which converged on strange set remained somewhat open.

We now apply the procedure outlined above to study the fluctuations in the L phase of the generalized Harper equation (12). We again make use of the explicit formulas of the Lyapunov exponents as obtained from Ref. [3]. Numerical iteration of the decimation equations shows that for $0 \leq \alpha < 1$, the fluctuations in the L phase are determined by the same strong coupling fixed point as for the Harper equation. For $\alpha \geq 1$, the decimation functions for the fluctuations in the L phase are found to flow away from the strong coupling fixed point. However, in analogy with the Harper case, the asymptotic behavior of the decimation functions appears to be independent of the value of λ throughout the L phase and is described by the same renormalization equations (10) and (11). That is, for all values of α, the decimation function f_n renormalizes to zero. Thus we can focus on the limit $\lambda \rightarrow \infty$, where E/λ tends to ± 2 for the band edges, and the generalized TBM for $\alpha \geq 1$ reduces to (ϕ is 1/2 for the negative and 0 for the positive band edge)

$$\{\pm 1 + \alpha\cos[2\pi\sigma(i - \tfrac{1}{2})]\}\eta_{i-1} = \alpha[1 - \cos(2\pi\sigma i)]\eta_i\,. \qquad (13)$$

Because of this simple TBM form (containing no parameter such as the energy which would be known to limited precision), the recursion relation (9) for the decimation function e_n can be iterated up to 35 times, thereby studying systems of the size 14 930 351. This accuracy is particularly crucial in order to observe higher order limit cycles. Our detailed numerical study reveals various limit cycles at certain values of α. Writing $\alpha = 1/\mathrm{abs}[\cos(2\pi r)]$, the limit cycles are observed for the rational values of r. For example, for $r = 1$, the period p of the limit cycle is found to be 3, while $p = 12$ for the values $r = 1/3, 1/6, 1/8, 1/14, 1/18, 2/9$. Furthermore, we observe $p = 24$ for $r = 1/7, 1/12, 1/16, 3/14, 3/16, 3/23$ and $p = 18$ for $r = 1/17, 1/19, 3/17, 3/19$. Since the RG equations cannot be iterated more than 35 times (because of memory limitations), we cannot see higher order cycles. Based on this study, we conjecture that for all rational values of r, the RG flow converges on a limit cycle of period which is a multiple of 3. However, the correlation between r and p still remains a mystery.

For arbitrary values of α, the decimation functions do not converge on a limit cycle. The plot of $e_{n+1}(0)$ vs $e_n(0)$ is found to converge on an invariant set which resembles an *orchid* flower (see Fig. 2). We conjecture that the invariant set of the renormalization operator is a universal strange attractor. The periodic orbits corresponding to the rational values of r are expected to be dense on the attractor. However, we conjecture that the probability to hit a periodic orbit is still zero. The confirmation of these ideas by an explicit solution of the renormalization equations remains open.

It turns out that the results at the strong coupling limit of the generalized Harper equation shed some light on the C phase of the model [12]. This is due to the fact that for a fixed value of α, the period of a strong coupling limit cycle coincides with the period of a similar limit cycle at the *critical* line $\lambda = \alpha$. Analogously, the existence of a strange set in the strong coupling limit strengthens our previous conjecture [4] on the existence of a similar set in the C phase.

In summary, our decimation studies show the existence of a new strong coupling renormalization fixed point which controls the universal fluctuations of the localized wave functions in the Harper equation. Unlike the trivial fixed point of the weak coupling limit, the strong coupling fixed point describes a new nontrivial universality class for the Harper equation. In analogy with the critical phase, these fluctuations are characterized by a universal scaling ratio which is determined by the value of the fixed point function at the origin. We are able to find a power series expansion of the fixed point and examine the stability of the fixed point under the renormalization. It turns out that a change in the NN or NNN coupling is an irrelevant perturbation as long as the NNN coupling is below a certain threshold. Above the threshold, the system is outside the basin of the attraction of the fixed point and the renormalization flow is attracted by an

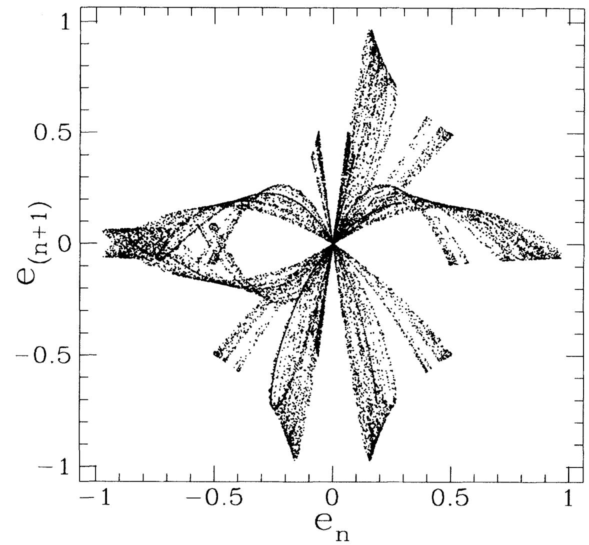

FIG. 2. A two-dimensional projection of the attractor of the renormalization flow in the strong coupling limit $\lambda \longrightarrow \infty$. For about 4000 different values of $\alpha > 1$, decimation equations were iterated 35 times and the first 10 iterates were ignored as transients.

invariant strange set of the function space. Our studies put the localized and the critical phase on the same footing. The localized phase could in fact be viewed as a fat critical phase. We hope that Bethe-ansatz tools [13], which were successfully applied to Harper critical point, will shed further light on the results described here.

The concept that the fluctuations in the random disorder problem may be fractal has been investigated recently [14–16]. The major problem in various numerical studies is that due to limited precision, it is not clear whether the studies really describe the properties of an infinite system. Our decimation equations, which take into account the internal frequency of the quasiperiodic system, are not suitable in the present form for studying models with random disorder. However, we investigated Eq. (1) where the cosine term was replaced by the unbounded potential $\tan(2\pi\sigma i)$, which has been shown to have the localization character of the Lloyd model describing a particle in a random potential [18]. Our renormalization study of this model (where the states are localized for all values of λ) showed that the fluctuations in the localized wave functions were self-similar and were described by the strong coupling fixed point of the model. These results put our study in a more general perspective and strengthen the likelihood of fractal characteristics in random or aperiodic systems exhibiting localization. Since localization is generic in a variety of systems [17] including the models of quantum chaos [18], the results described here open a new avenue in the localization theory.

The research of I. I. S. is supported by National Science Foundation Grant No. DMR 093296. J. A. K. has been supported by the Academy of Finland under Contract No. 4385 and by the Niilo Helander Foundation. J. A. K. is also grateful for the hospitality of the Research Institute for Theoretical Physics in Helsinki.

*Electronic address: isatija@sitar.gmu.edu

[1] P. G. Harper, Proc. Phys. Soc. London A **68**, 874 (1955).
[2] For a review, see J. B. Sokoloff, Phys. Rep. **126**, 189 (1985).
[3] J. H. Han, D. J. Thouless, H. Hiramoto, and M. Kohmoto, Phys. Rev. B **50**, 11 365 (1994); Y. Hatsugai and M. Kohmoto, Phys. Rev. B **42**, 8282 (1990).
[4] J. A. Ketoja, I. I. Satija, and J. C. Chaves, Phys. Rev. B **52**, 3026 (1995).
[5] J. A. Ketoja and I. I. Satija, Phys. Lett. A **194**, 64 (1994); Physica (Amsterdam) (to be published).
[6] S. Ostlund and R. Pandit, Phys. Rev. B **29**, 1394 (1984); S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, Phys. Rev. Lett. **50**, 1873 (1983).
[7] S. Aubry and G. André, Ann. Israel Phys. Soc. **3**, 133 (1980).
[8] The fluctuations for $i < 0$ satisfy the same TBM except that η_{i+1} is multiplied by λ and η_{i-1} by $1/\lambda$. This causes the fluctuations to be symmetric around $i = 0$.
[9] J. A. Ketoja, Phys. Rev. Lett. **69**, 2180 (1992).
[10] The decimation equation here is written in a slighly different form from our previous papers [5] in order to have bounded asymptotic limits of the decimation functions e_n and f_n as $n \longrightarrow \infty$, also in the localized case $\lambda > 1$.
[11] If the absolute value is not taken, the fixed point is observed only at the positive band edge. The negative band edge corresponds to the 3-cycle $[-e^*(-\sigma x), e^*(x)]$, $[e^*(-\sigma x), -e^*(x)]$, and $[-e^*(-\sigma x), -e^*(x)]$.
[12] Because of limited precision in the energy, the decimation equations can be iterated only about 28 times in the C phase. However, in the strong coupling limit it is possible to go up to 35 iterations.
[13] P. B. Wiegmann and A. V. Zabrodin, Phys. Rev. Lett.**72**, 1890 (1994); L. D. Faddeev and R. M. Kashaev, Commun. Math. Phys. **169**, 181 (1995).
[14] L. Pietronero, A. P. Siebesma, E. Tosatti, and M. Zannetti, Phys. Rev. B **36**, 5635 (1987).
[15] M. Schreiber and H. Grussbach, Phys. Rev. Lett. **67**, 607 (1991).
[16] F. Lema and C. Wiecko, Phys. Scr. **47**, 129 (1993); C. Wiecko and E. Roman, Phys. Rev. B **30**, 1603 (1984).
[17] *Proceedings of International Conference on Localization held at Imperial College, London 1990,* edited by K. A. Bebedict and J. T. Chalker (Institute of Physics, New York, 1990).
[18] S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).