Comment on "Superinstantons and the Reliability of Perturbation Theory in Non-Abelian Models"

In a recent Letter [1] Patrascioiu and Seiler argued that the results of standard perturbation theory (PT) are not valid for two-dimensional models with a non-Abelian global symmetry. They considered such a theory in a finite box of size L, with special boundary conditions (BC) called "superinstantons" (SI), and showed that in the thermodynamic limit $L \to \infty$ the 2-loop corrections are finite, but differ from those obtained from PT with standard BC. They concluded that, when SI configurations are taken into account, renormalization group (RG) β functions are modified, and that the limit $L \to \infty$ and the weak-coupling expansion do not commute.

In fact, the results of [1] do not contradict standard PT. Indeed one can show from general principles the following: (1) PT with a SI BC is infrared (IR) divergent; (2) the IR divergent items are associated, via the short distance operator product expansion (OPE), with singular local operators, not present for classical backgrounds; (3) taking into account these operators, the perturbative RG functions are not modified.

Let us show this for the nonlinear $O(N)$ σ-model considered in [1], with N-component unit vector field $\vec{S} = (\vec{\pi}, \sigma = \sqrt{1 - \vec{\pi}^2})$, defined in a square box $\Lambda_L = [-L/2, L/2] \times [-L/2, L/2]$, with Dirichlet ($D$) BC $\vec{\pi} = \vec{0}$ on $\partial\Lambda_L$, and with the additional SI constraint $\vec{\pi}(0) = \vec{0}$ at the origin. The 2-point function with the SI BC is related to that with the D BC by

$$\langle \vec{S}(x) \cdot \vec{S}(y) \rangle_{\mathrm{SI}} = \frac{\langle \vec{S}(x) \cdot \vec{S}(y) \delta(\vec{\pi}(0)) \rangle_D}{\langle \delta(\vec{\pi}(0)) \rangle_D}, \qquad (1)$$

with $\delta(\vec{\pi})$ the Dirac distribution in $\mathbb{R}^{N-1}$. Its perturbative expansion is $\langle \vec{S}(x) \cdot \vec{S}(y) \rangle_{\mathrm{SI}} = 1 + c_1 g + c_2 g^2 + \cdots$. For simplicity we first regularize the short-distance divergences by using dimensional regularization, with space dimension $D = 2 - \epsilon$ ($\epsilon > 0$), and use the continuum action $S = (1/2g) \int d^D x [\partial \vec{S}(x)]^2$. Adapting the results of [2], and the techniques of [3,4] to deal with the singular operator $\delta(\vec{\pi})$, it is easy to show that the $L \to \infty$ expansion of Eq. (1) is given by a sum over local operators $A(0)$ located at $x = 0$, and with support in field space at $\{\vec{\pi}(0) = 0\}$,

$$\langle \vec{S}(x) \cdot \vec{S}(y) \rangle_{\mathrm{SI}} = \sum_A C^A(x, y) \frac{\langle A(0) \rangle_D}{\langle \delta(\vec{\pi}(0)) \rangle_D}, \qquad (2)$$

where the OPE coefficients C^A are *independent* from the specific BC used and from L, and are *finite* in PT. The BC and L dependence is contained entirely in the expectation value ratios (evaluated in the box Λ_L with D BC) $\langle A(0) \rangle_D / \langle \delta(\vec{\pi}(0)) \rangle_D$, which scale as $L^{-\dim[A]} f_A(g L^\epsilon)$, with f_A calculable in PT, and where $\dim[A]$ is the canonical dimension of A ($\dim[\vec{\pi}] = 0$, $\dim[x] = -1$). Only operators with $\dim[A] = 0$ can give finite or divergent contributions when $L \to \infty$; operators with $\dim[A] > 0$ give subdominant corrections. The first dimensionless operator is $A_0 = \delta(\vec{\pi})$: $f_{A_0 = 1}$, and the coefficient $C^{A_0} = 1 + c_1^{(0)} g + c_2^{(0)} g^2 + \cdots$ gives the standard IR finite PT. However, additional operators appear, of the form $A_n = (\Delta_{\vec{\pi}})^n \delta(\vec{\pi})$, with $\Delta_{\vec{\pi}}$ the Laplacian in $\mathbb{R}^{N-1}$. The first one, A_1, is such that $C^{A_1} = c_2^{(1)} g^2 + c_3^{(1)} g^3 + \cdots$, and that $f_{A_1}(g) = f_{-1}^{(1)} g^{-1} + f_0^{(1)} + f_1^{(1)} g + \cdots$. Simple diagrammatics shows that $C^{A_n} = \mathcal{O}(g^{2n})$ and $f_{A_n} = \mathcal{O}(g^{-n})$. Therefore, the coefficient of g of Eq. (1) behaves as $c_1 = c_1^{(0)} + L^{-\epsilon} c_2^{(1)} f_{-1}^{(1)} + \cdots$ when $L \to \infty$. Its IR limit coincides with the standard PT result $c_1^{(0)}$. The coefficient of g^2 behaves as $c_2 = c_2^{(0)} + c_2^{(1)} f_0^{(1)} + L^{-\epsilon} c_3^{(1)} f_{-1}^{(1)} + \cdots$ and has a finite IR limit, different from the PT result $c_2^{(0)}$. However, the coefficient of g^3 is IR singular $c_3 = c_2^{(1)} f_1^{(1)} L^\epsilon + \cdots$, as well as the higher order terms. The existence of these IR divergences is generic, except for the Abelian $O(2)$ model, where one can show that they vanish identically.

These conclusions are independent of the regularization, and can be extended to the $D = 2$ lattice model of [1]: With SI BC, the 2-point function is IR finite at order g^2 but differs from standard PT; at order g^n, $n > 2$ (it is IR divergent) with $\ln(L)^{n-2}$ terms. These results are valid *order by order* in PT, and apply to the renormalized theory as well: to construct the continuum limit for finite L, besides the coupling constant and wave-function renormalization, one must also renormalize the SI insertion operator $A_0(0)$. Taking this effect into account, the PT β functions are unchanged, but the renormalized theory with SI BC is IR divergent at order g^3 and beyond. Similar problems are expected to occur for the non-Abelian gauge theories considered in Ref. [5].

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