

Skyrme Crystal in a Two-Dimensional Electron Gas

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The ground state of a two-dimensional electron gas at Landau level filling factors near $\nu = 1$ is a Skyrme crystal with long-range order in the positions and orientations of the topologically and electrically charged elementary excitations of the $\nu = 1$ ferromagnetic ground state. The lowest energy Skyrme crystal is a square lattice with opposing postures for topological excitations on opposite sublattices. The filling factor dependence of the electron spin polarization, calculated for the square lattice Skyrme crystal, is in excellent agreement with recent experiments.

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The incompressible [1] ground state of a two-dimensional electron gas (2DEG) at Landau level filling factor $\nu = 1$ is ferromagnetic. (Here $\nu \equiv N/N_\phi$ is the ratio of the number of electrons to the orbital degeneracy of a Landau level; $N_\phi = AB/\Phi_0 = A/2\pi\ell^2$, where A is the area of the system, Φ_0 is the magnetic flux quantum, and B is the magnetic field strength.) Recently, it has been shown [2–5] that, in the limit of weak Zeeman coupling, the lowest energy charged excitations of this state are spin textures known as *Skyrmions* [6]. Skyrmions are the lowest energy topologically charged spin-texture excitations of the SU(2) nonlinear σ (NL σ) model which describes the long wavelength properties of isotropic ferromagnets. The equivalence of physical charge and topological charge in the present system is a consequence [2,5] of the quantum Hall effect [1] and is responsible for the dominating role of Skyrmions in determining many physical properties.

Pioneering studies of Skyrmions in the quantum Hall regime relied on NL σ models generalized [2,5] to account for Zeeman coupling to the spin and for the Hartree (electrostatic) interactions present when the charge density is nonuniform. States in the NL σ model are specified by a space-dependent unit vector which describes the local orientation of the spin magnetic moment. A Skyrmion is characterized by the sign of its topological charge, by its size, and by the global orientation of the spin; with no Zeeman or Hartree coupling, the energy of an isolated Skyrmion is independent of all three. When Zeeman coupling is included the spin moment outside a Skyrmion aligns with the field ($m_z = 1$), the spin moment at the center of a Skyrmion is oriented in opposition to the field ($m_z = -1$). The perpendicular component (m_\perp) of the spin moment, which must be nonzero as m_z changes from -1 to 1 , has a vortex at the Skyrmion center. The global azimuthal orientation of m_\perp is, importantly for the work described here, still arbitrary. Because the spin moment is reversed in the interior

of a Skyrmion, Zeeman coupling favors small Skyrmions. On the other hand, the Hartree self-interaction energy of the Skyrmion favors large Skyrmions so that an optimal Skyrmion size is established [2]. For Zeeman coupling strengths typical of physical systems the estimated Skyrmion size is comparable to microscopic lengths, motivating a microscopic approach. Recently [7], using a microscopic Hartree-Fock approximation, we confirmed the main predictions of the field-theory approach and obtained an estimate of the number of reversed spins for each (Skyrmion) charge added to the system at $\nu = 1$. The quantitative agreement between this estimate and subsequent Knight shift measurements [8] appears to provide incontrovertible evidence in favor of the exotic topological nature of the low-energy charge carriers in this system. In this Letter we report on a Hartree-Fock approximation study of the ground state of the 2DEG for ν near 1, when interactions between the charged spin textures, which we will refer to as Skyrmions even when distorted by Zeeman and Hartree couplings, become important.

The most important result of our calculation is shown in Fig. 1, where we compare our theoretical results for the spin-polarization P of the two-dimensional electron gas as a function of filling factor with experiment [8]. (P is the spatial average of m_z .) Without interactions the spins are completely aligned by the Zeeman field for $\nu < 1$. The polarization is gradually reduced for $\nu > 1$, reaching zero for $\nu = 2$, since the Pauli exclusion principle forces the occupation of reversed spins. The large slope of P on either side of $\nu = 1$ is in sharp contrast with the noninteracting electron result and can be related to the number of reversed spins in an isolated Skyrmion [2,7,8]. Our calculations at finite $|\nu - 1|$ were performed assuming that the ground state of the system is a crystal of charged Skyrmion spin textures. These states are analogous to the Skyrme crystal states which arise [9] in studies of dense nuclear matter using Skyrme's topological excitation model. The Hartree-Fock

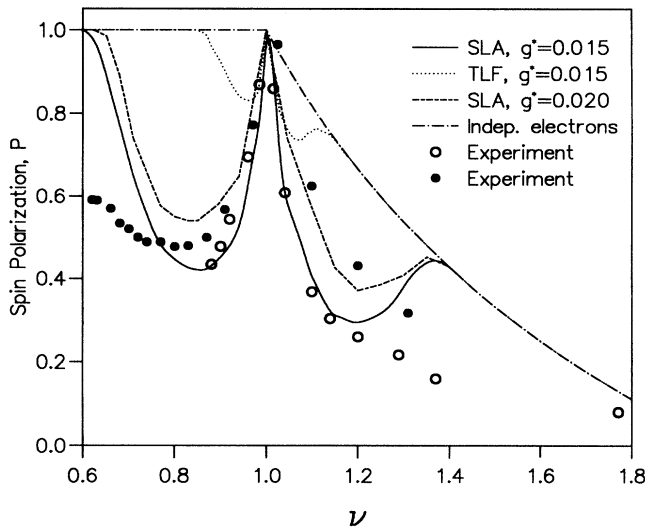


FIG. 1. Variation of the spin polarization P as a function of ν for SLA and TLF Skyrme crystal states and for different values of $g^* \equiv g\mu_B B/(e^2/\ell)$. The open circles are experimental data obtained for g^* near 0.016, while the filled circles show data obtained for g^* near 0.021. The spin polarization in the WC state is the same as that for independent electrons.

equations have many different solutions; all have long-range order both in the spatial arrangement of Skyrmion centers *and* in the azimuthal orientation of m_\perp near each Skyrmion center. Results are shown for two cases: (i) a triangular lattice of Skyrmions with identical orientations of m_\perp near each site (TLF state) and (ii) a square lattice with oppositely directed orientations of m_\perp on the two square sublattices (SLA state). (The initials A and F in the labels of these states suggest the obvious analogies with ferromagnetic and antiferromagnetic states of spin systems on two-dimensional square and triangular lattices.) We see in Fig. 1 that the slope of the spin-polarization curve decreases in magnitude as we move away from $\nu = 1$ in both cases, indicating that the Skyrmions shrink as they become more dense. The shrinking is evidently much more rapid for the TLF state than for the SLA state. The results for P for the latter case are in quite reasonable agreement with experiment and indeed, as we discuss below, we find that the square Skyrme lattice has lower energy than the triangular Skyrme lattice except at very small $|\nu - 1|$. Comparison between theory and experimental results for P suggests that the ground state is a square Skyrme crystal until $|\nu - 1| \approx 0.2$.

Our calculations are greatly simplified by working entirely within the lowest orbital Landau level, and the Green's function equation of motion approach we have developed previously is readily adapted to the present situation [10]. For small enough $|\nu - 1|$, the Skyrmions can be considered to be point particles, and it is known

from Madelung energy calculations for 2D lattices [11] that they will form a triangular lattice. Nevertheless, we find that by the time $\nu = 1.1$ the SLA state is lower than the TLF in energy by ≈ 30 times the amount by which the Madelung energy would favor the opposite energetic ordering of the states. The self-consistent spin textures for the TLF and SLA states at $\nu = 1.1$ are illustrated in Fig. 2. The Skyrmions are evidently more compact for the higher energy TLF state, and this is the origin of the differences in spin polarization between these states.

To understand the differences between SLA and TLF states in greater depth it is useful to compare the contributions to the energy per electron from exchange, Hartree, and Zeeman energies as shown in Fig. 3 for $g\mu_B B = 0.015e^2/\ell$. We express our results in terms of the energy (ϵ_{qp}) increase per quasiparticle, $N_{qp} = |N - N_\phi|$, when the quasiparticles are created at fixed N by varying N_ϕ :

$$\epsilon_{qp} \equiv \frac{\nu[\epsilon + (e^2/\ell)\sqrt{\pi/8} + g\mu_B B/2]}{|1 - \nu|}, \quad (1)$$

where $\epsilon = E/N$ is the energy per electron, g is the g factor for the 2DEG system, and μ_B is the electron Bohr magneton. [The Hartree, exchange, and Zeeman contributions to ϵ_{qp} are given by terms of corresponding origin on the right-hand side of Eq. (1); $-(e^2/\ell)\sqrt{\pi/8}$ is the exchange energy per particle at $\nu = 1$.] This convention for the energy per charge is implicit in the field theoretical expressions, and we choose it here because it will allow our Hartree-Fock results and results obtained in the field-theory description to be compared directly. In the limit of vanishing Skyrmion density ϵ_{qp} is the so-called neutral quasiparticle energy [12]. It follows from particle-hole symmetry that with this definition anti-Skyrmion quasiparticle energies are smaller than Skyrmion quasiparticle energies by $g\mu_B B$.

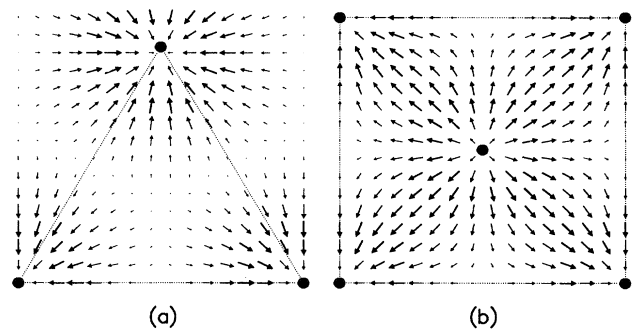


FIG. 2. Two-dimensional vector representation of the x - y components of the spin density of a crystal of Skyrmions as obtained in the HFA. The length of the arrows is proportional to the local magnitude of m_\perp , and their direction indicates the local orientation. The magnitude of m_\perp is small for m_z near -1 , near the center of a Skyrmion, and for m_z near $+1$, outside of a Skyrmion. (a) TLF state at $\nu = 1.1$ and $g^* = 0.015$; (b) SLA state at $\nu = 1.1$ and $g^* = 0.015$.

We first discuss the constituents of the energy for the Wigner crystal (WC) state which occurs near $\nu = 1$ in the limit of very strong Zeeman coupling. In this state a triangular (Wigner) crystal [13] of single-reversed-spin quasiparticles is formed on the background of a full Landau level of aligned spins. The reversed spins are localized to the greatest extent possible with the lowest Landau level. For the WC state, the exchange and Hartree contributions to the energy per quasiparticle, ϵ_{qp}^X and ϵ_{qp}^H , can be calculated analytically [14] in the limit of small ν_{qp} : $\epsilon_{qp}^X \rightarrow (e^2/\ell)[(\pi/8)^{1/2} - (\pi/16)^{1/2}] \approx 0.1835(e^2/\ell)$ and $\epsilon_{qp}^H \rightarrow (e^2/\ell)(\pi/16)^{1/2} \approx 0.4431(e^2/\ell)$, the Hartree self-interaction energy of a lowest Landau level quantized cyclotron orbit. The exchange energy, which is minimized when all spins are parallel, is nearly constant with increasing ν_{qp} until the quasiparticles begin to overlap significantly, while the Hartree energy is reduced at larger ν_{qp} due to spatial correlations of the quasiparticles. When the size of the quasiparticles ($\sim \ell$ for the WC state) is much smaller than the distance between quasiparticles the Hartree energy is reduced by the Madelung energy per particle [11]; when the quasiparticles form a triangular lattice $\epsilon_{Mad} \approx -0.7821\nu_{qp}^{1/2}(e^2/\ell)$. In Fig. 3 we have plotted the difference between the Hartree energy and the Madelung energy vs ν_{qp} . For the WC state the

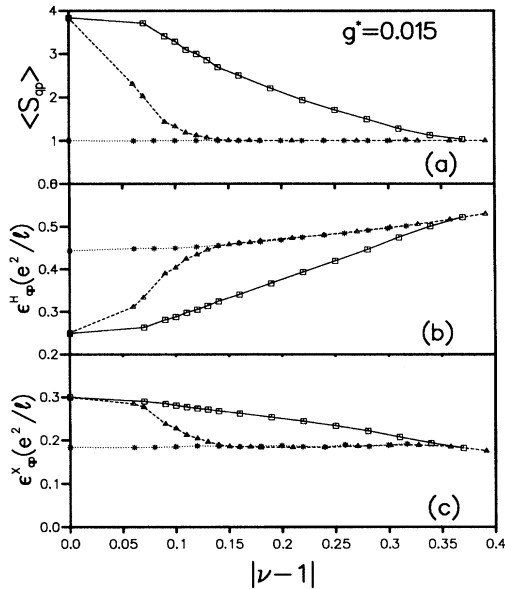


FIG. 3. Zeeman (a), Hartree (b), and exchange (c) contributions to the energy per quasiparticle for the Wigner crystal state (*) and for the TLF (triangles) and SLA (squares) Skyrme crystal states. The results for $\nu = 1$ were obtained analytically for the WC state and from isolated Skyrmion calculations for the Skyrme crystal states. The Zeeman energy is expressed in terms of the average number of reversed spins per quasiparticle and the Madelung contribution has been subtracted from the Hartree contribution, as discussed in the text.

remaining variation is due to the fact that the electrons are localized in cyclotron orbits and not at points. The Zeeman energy contribution to ϵ_{qp} is $g\mu_B B \langle S_{qp} \rangle$, where $\langle S_{qp} \rangle$ is the number of reversed spins per quasiparticle. ($\langle S_{qp} \rangle$ is smaller for anti-Skyrmions than for Skyrmions by 1.) For the WC state $\langle S_{qp} \rangle \equiv 1$ for Skyrmions and 0 for anti-Skyrmions.

In the absence of Zeeman coupling a Skyrmion quasiparticle has lower energy than a reversed spin quasiparticle because its charge is spread out over an arbitrarily large distance, reducing ϵ_{qp}^H to zero while at the same time keeping spins locally nearly parallel so that ϵ_{qp}^X increases only [2,5] to $(e^2/\ell)(\pi/32)^{1/2} \approx 0.3133(e^2/\ell)$. When a finite Zeeman coupling is present, the Skyrmion quasiparticle must shrink to reduce its Zeeman energy [2,7] and the Hartree energy increases. For an isolated Skyrmion ($\nu_{qp} \rightarrow 0$) and $g\mu_B B = 0.015e^2/\ell$ we find, using the approach [15] of Ref. [7], that $\epsilon_{qp}^H = 0.2499(e^2/\ell)$, $\epsilon_{qp}^X = 0.2957(e^2/\ell)$, and $\langle S_{qp} \rangle = 3.83$. In Fig. 3 we see that for the TLF state the Hartree energy quickly rises from the isolated Skyrmion value toward the WC value and $\langle S_{qp} \rangle$ quickly decreases from the isolated Skyrmion value to 1 as ν_{qp} increases. Evidently, there is a strong short-range repulsive interaction between Skyrmions in the TLF Skyrme state which quickly shrinks them into single-reversed-spin quasiparticles. The origin of this repulsion is an increase in exchange energy in the region between neighboring Skyrmions with identical azimuthal orientations of m_\perp . As we see in Fig. 2 the azimuthal orientation of m_\perp must then change by π relatively abruptly along the line connecting neighboring Skyrmion centers; in the field-theoretical description [2,5] the associated increase in exchange energy is described by a term $\propto |\nabla m_\perp|^2$ and the resulting repulsive interaction will be roughly proportional to m_\perp^2 evaluated at the midpoint between Skyrmion centers. Clearly this repulsive interaction is minimized if neighboring Skyrmions have opposite azimuthal orientations for m_\perp as illustrated for the SLA state in Fig. 2. (It seems clear that the lowest energy triangular Skyrme lattice will be a three-sublattice state with $2\pi/3$ relative orientation but this state will, we believe, be higher in energy than the SLA state unless ν is very close to 1.) The exchange energy thus favors bipartite lattices for a Skyrme crystal. Our numerical results for the SLA state show that the Skyrmions shrink relatively slowly as ν_{qp} increases; for this arrangement the Skyrme crystal is able to tolerate much tighter packing of the Skyrmions, and the rapid decrease in spin polarization persists out to quite large values of ν_{qp} as shown in Fig. 1 and observed experimentally [8].

The size of a quasiparticle in a Skyrme crystal is limited by the lattice constant of the crystal until, with decreasing ν_{qp} , the lattice constant becomes substantially larger than the size of an isolated Skyrmion quasiparticle. For weaker Zeeman couplings, isolated Skyrmions are larger [2,7] and spin polarization will decrease more rapidly for ν very

close to $\nu = 1$; for larger values of $|\nu - 1|$, where the Skyrmion size is limited by inter-Skyrmion interactions, the spin polarization is much less dependent on the Zeeman coupling. For example, we find for the TLF state at $g\mu_B B/(e^2/\ell) = 0.0015$ that $P = 0.7$ at $\nu = 1.03$, substantially smaller than the values reported in Fig. 1, while for $\nu = 1.15$, $P = 0.7$ again, nearly identical to Fig. 1.

From our calculations we can conclude that for typical $g\mu_B B$ values, the ground state of the 2DEG near $\nu = 1$ is composed of Skyrmionic charged excitations of the ferromagnetic $\nu = 1$ background. When these quasiparticles are sufficiently dilute they will crystallize because of the absence of kinetic energy. Although we have not completed an exhaustive survey of possible Skyrme lattices, we believe that SLA Skyrme crystal will be lowest in energy once the lattice constant is small enough for the gradient energy to overwhelm the relatively modest structural preferences of the Hartree energy. It is difficult for us to estimate the range of filling factors over which the Skyrme crystal is favored over fluid states of the 2DEG [16]. However, it seems unlikely that the energy competitions which determine Skyrmion size will be very sensitive to the existence of long-range order. We anticipate the existence of a Skyrme liquid state, which will be lower in energy than maximally spin-polarized liquid states and which will have a spin polarization quite close to that of the SLA state. The Skyrme liquid should have a typical coordination number closer to 4 than to 6 because of the dependence of the exchange energy on the relative azimuthal orientations of m_\perp for nearby Skyrmions.

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- [15] Since the \hat{z} component of the total spin is a good quantum number, for an isolated Skyrmion $\langle S_z \rangle$ is an integer in the exact many-body ground state. The Hartree-Fock estimate is a continuous function of system parameters and applies unless $k_B T$ is smaller than energetic separation between single-Skyrmion states with adjacent values of $\langle S_z \rangle$. For typical systems we do not expect the discreteness of the allowed values for $\langle S_z \rangle$ to be observable above ~ 1 K. In the Skyrme crystal Skyrmion-Skyrmion interactions will cause $\langle S_z \rangle$ to fluctuate on each site even at $T = 0$.
- [16] By comparing energies, we know that for $g\mu_B B = 0.015e^2/\ell$ the Laughlin fluid states at $\nu = 2/3$ and $\nu = 4/3$ are lower in energy than the SLA Skyrme crystal. For $\nu = 4/5$ and $\nu = 6/5$, the SLA Skyrme crystal has lower energy than the Laughlin states. This, however, does not exclude the possibility of an unpolarized Laughlin liquid or a Skyrme liquid state at these filling factors.