Emergence of Quenched Phases and Second Order Transitions for Sums of Multifractal Measures

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It is shown that superpositions of multifractal measures provide an ubiquitous mechanism for nonanalytic behavior of characteristic thermodynamic quantities. We find first and second order phase transitions. The latter frequently show up as experimentally observable stopping points in $f(\alpha)$ curves. Our results are derived analytically for sums of multiplicative and Markovian measures. The critical exponents of the continuous transition define a new universality class of systems, which include equivalent Ising models with long-ranged multispin interactions.

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Singular continuous measures have long been considered as exotic mathematical constructs. Since the success of nonlinear dynamics [1] and the theory of fractals [2–4] they became popular as multifractal measures [5], which characterize strange attractors, fractal aggregates, turbulent velocity fields, etc. Further, the role of singular continuous measures and their analysis in terms of multifractals was emphasized in very diverse fields such as x-ray scattering from aperiodic crystals [6], random-field Ising models [7], and, most recently, for problems of anomalous quantum diffusion and quantum chaos [8]. The well-known formal equivalence of a multifractal analysis to statistical mechanics implies the existence of phase transitions, to be understood as nonanalytic behavior of appropriate thermodynamic quantities [9]. The purpose of this Letter is to establish a new, experimentally relevant, and universal mechanism for second order phase transitions in the thermodynamics of multifractals.

We address experiments where for some reason the full multifractal structure of the object of interest is not accessible, but certain projections can be measured. Examples are the observation of clouds, images of fractal structures in the Universe, or, quite general, of fractal structures with dimensions larger than that of the image. Similar problems occur if dynamically generated attractors are reconstructed from time series with too small embedding dimensions. The implications of such circumstances for a multifractal analysis constitute an unsolved theoretical problem [10]. A related situation is obtained if one observes the superposition of a finite or countable infinite number of independent multifractals with measure. In scattering problems and for spectral measures such sums arise if one has independent contributions to the spectrum, e.g., from localized regions far apart in coordinate or phase space. Typically, the support of the singular distributions that add up are overlapping. Very little is known for this generic case. In the following we treat the limit of large overlap where the support of all contributing distributions are identical.

We present our results for three analytically solvable classes, where in each model class a qualitatively new feature becomes apparent.

(a) Superpositions of multiplicative Cantor measures.— The most elementary case is obtained by adding M complete, self-similar, multiplicative measures $\mu(\nu)$ supported by the same monoscale Cantor set with contraction ratio $l \leq 1/2$. This means one considers the measure $\mu = \sum_{\nu=1}^{M} \pi(\nu)\mu(\nu)$ with weights $\pi(\nu)$ $[\sum_\nu \pi(\nu) = 1]$. At the Nth level of hierarchical construction (binary tree) of a multiplicative component $\mu(\nu)$ one finds 2^N "boxes" of length l^N, where $\binom{N}{j}$ boxes have a "mass" content equal to $p_0(\nu)^j p_1(\nu)^{N-j}$. The quantities $p_0(\nu)$ and $p_1(\nu) = 1 - p_0(\nu)$ are the mass distribution ratios for each step in this construction and j is the number of digits "zero" in the binary address of the fractal elements (see, e.g., [3]). For the sum μ the mass content of such a box is accordingly given by

$$P_N(j) = \sum_{\nu=1}^{M} \pi(\nu) p_0(\nu)^j p_1(\nu)^{N-j}. \tag{1}$$

The scaling behavior of multifractals is captured by $\tau(q)$, the growth rate of the Nth level partition function $Z_N(q)$, i.e., $\tau(q) \equiv \lim_{N\to\infty}(N \ln l)^{-1} \ln Z_N(q)$ with $Z_N(q) = \sum_{j=0}^{N} \binom{N}{j} P_N(j)^q$ [3,5,11]. In the large N limit and with $\xi = j/N$ one finds $Z_N(q) \sim \int_0^1 e^{-N[qa(\xi)-s(\xi)]}\,d\xi$. The entropy term $s(\xi) = -\xi \ln \xi - (1-\xi)\ln(1-\xi)$ is due to the binomial coefficient and $a(\xi) \equiv -\lim_{N\to\infty} N^{-1} \ln P_N(\xi N)$. By the usual saddle point argument $\tau(q)$ is obtained as

$$\tau(q) = \min_{\xi} g(\xi; q)/[\ln l], \tag{2a}$$

$$g(\xi; q) = q \min_{\nu} a_\nu(\xi) - s(\xi), \tag{2b}$$

where in Eq. (2b) one is aware that the largest term in Eq. (1) dominates in the limit $N \to \infty$ and that therefore the box contents scale with $a(\xi) = \min_\nu a_\nu(\xi)$ with $a_\nu(\xi) = -\xi \ln p_0(\nu) - (1-\xi)\ln p_1(\nu)$. In Fig. 1 we plot $g(\xi; q)$ for the superposition of $M = 2$ measures $\mu(\nu)$ and for two parameter sets. One finds $a(\xi) = a_2(\xi)$ for $\xi \leq \xi_c = \{1 + \ln[p_0(2)/p_0(1)]/\ln[p_1(1)/p_1(2)]\}^{-1}$ and $a(\xi) = a_1(\xi)$ for $\xi \geq \xi_c$ [we assume $p_0(1) > p_0(2)$]. Therefore there exist exactly three possibilities: Either the

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minimum $\xi(q)$ of $g(\xi;q)$ is provided by the unique minimum of one of the two functions $g_\nu(\xi;q) = q\alpha_\nu(\xi) - s(\xi)$, $\nu = 1, 2$, or it lies at their intersection point ξ_c (vertical line in Fig. 1). Correspondingly, one finds that in the two former cases $\tau(q)$ is determined by one of the contributing measures with the well-known result $\tau(q) = \tau_\nu(q) = \ln[p_0(\nu)^q + p_1(\nu)^q]/\ln l$, $\nu = 1$ or 2, whereas in the third case a new phase arises with $\tau(q) = \tau_0(q) \equiv [q a(\xi_c) - s(\xi_c)]/|\ln l|$. The latter is appropriately called a *quenched* phase, since it is characterized by a vanishing susceptibility: Applying a small "field" h; i.e., adding a term $h\xi$ to $g(\xi;q)$ does not alter the value of $\xi(q)$, which is locked to ξ_c. One always observes a first order transition between phase 1 and phase 2 at $q_{1\mathrm{st}} = 1$. For certain parameters [as in Fig. 1(b)] we find, in addition, a second order phase transition from phase $\nu = 1$ or 2 into the quenched phase 0 as q gets smaller than the negative critical value $q_{2\mathrm{nd}} = -\ln\frac{\ln[p_0(1)/p_0(2)]}{\ln[p_1(2)/p_1(1)]} \Big/ \ln[p_0(\nu)/p_1(\nu)]$ [$\nu = 1$ in Fig. 1(b)].

It is a *geometrical* relationship between the multifractal components that accounts for the second order transition and the existence of the quenched phase. This is best understood by considering the entropic quantity $f(\alpha)$, the Legendre transform of $\tau(q)$ [5], which is also measured in most experiments. In Fig. 2(a) the $f(\alpha)$ curve corresponding to the case of Fig. 1(b) is shown together with the curves for the isolated components. As usual the first order phase transition is detected as a straight line segment on the bisectrix (dashed) connecting the curves for the two components [12]. The second order transition is quite drastically manifest as a stopping point at the intersection of the curves $f_1(\alpha)$ and $f_2(\alpha)$ corresponding to the isolating measures $\mu(1)$ and $\mu(2)$. Accordingly the $f(\alpha)$ curve stops at $\alpha = \alpha_{\max}$ with *finite* left derivative [13]. In contrast, for the parameters of Fig. 1(a), which are chosen to yield the *same* curves $f_\nu(\alpha)$, the $f_\nu(\alpha)$ spectrum for the sum $\mu = \pi(1)\mu(1) + \pi(2)\mu(2)$ smoothly continues across the intersection point on the lower dotted line. For $\alpha < \alpha_{\max}$ the spectra for the two cases are identical. Obviously this difference cannot be explained merely by form and location of the functions $f_\nu(\alpha)$. The explanation follows from Fig. 2(b), where we plotted [for the case of Fig. 1(b)] the scaling exponents $\alpha_\nu(\xi) = a_\nu(\xi)/|\ln l|$ of the measures $\mu(\nu)$ restricted to the fractal subset $S(\xi)$ of the support S, and $f(\xi) = s(\xi)/|\ln l|$, the fractal dimension of this subset [3]. We see that the superposition is such that the piecewise linear function $\alpha(\xi) = \min_\nu \alpha_\nu(\xi)$ exhibits a maximum at the intersection point ξ_c. The picture for the case without continuous transition [Fig. 1(a)] differs only in so far as one of the curves $\alpha_\nu(\xi)$ is replaced by its mirror image with respect to the line $\xi = 0.5$ resulting in a monotonous function $\alpha(\xi)$. From the functions $\alpha_\nu(\xi)$ and $f(\xi)$ the value $f(\alpha)$ for some α in the range of $\alpha(\xi)$ is found by taking the preimage ξ of α and reading off the corresponding value $f(\xi)$. If there are two preimages [as in Fig. 2(b)], the value with the larger $f(\xi)$ has to be taken [14]. This construction explains the absence of a continuous transition in the case of Fig. 1(a), and also why in its presence the identities $\alpha_{\max} = \tau_0'(q) = \alpha_1(\xi_c) = \alpha_2(\xi_c)$ and $f(\alpha_{\max}) = f_1(\alpha_{\max}) = f_2(\alpha_{\max}) = f(\xi_c) > 0$ are fulfilled. Thus, *the nonmonotonicity of $\alpha(\xi)$ is the reason for the second order phase transition.* Consequently, it is found only if one of the functions $\alpha_\nu(\xi)$ is decreasing while the other is increasing. For monoscale Cantor measures this is true inside the rectangle $[1/2 \le p_0(1) \le 1] \times [0 \le p_0(2) \le 1/2]$, which covers half the available, symmetry reduced parameter space, the triangle $0 \le p_0(2) \le p_0(1) \le 1$. The mech-

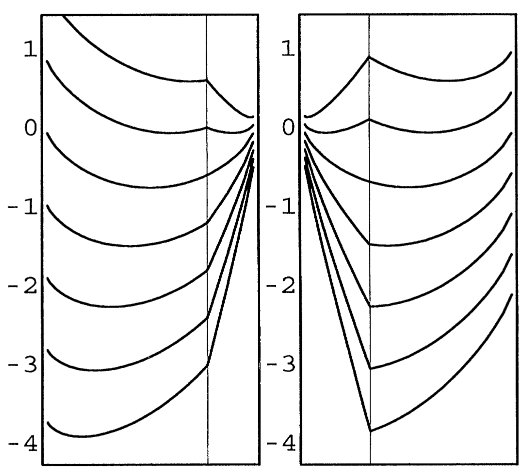

FIG. 1. The "free energy functional" $g(\xi;q)$ for integer q values between $q = 2$ (top) and $q = -4$ (bottom) for the superposition of two multifractal Cantor measures [(a) $p_0(1) = 0.9$, $p_0(2) = 0.6$, (b) $p_0(1) = 0.6$, $p_0(2) = 0.1$]. In both cases a jump of the minimum $\xi(q)$ of $g(\xi;q)$ at "inverse temperature" $q = 1$ (second curve from top) results in a first order transition. In (b) a second order transition is observed as $\xi(q)$ gets locked in a continuous way to ξ_c (thin vertical line) as q decreases below the critical value $q_{2\mathrm{nd}} = -1.955\ldots$ (lowest three curves).

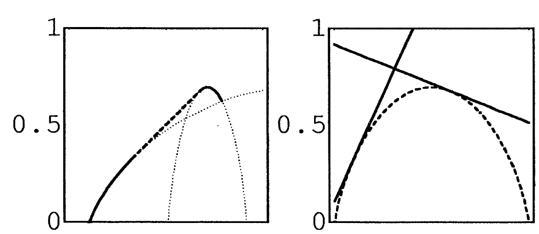

FIG. 2. (a) $f(\alpha)$ curves for the superimposed (full line) and for the separate measures (dotted lines) in the presence of the continuous transition, and (b) its explanation by the functions $\alpha_\nu(\xi)$, $\nu = 1, 2$ (full lines) and $f(\xi)$ (dashed). The continuous transition of Fig. 1(b) appears as a stopping point in the $f(\alpha)$ curve.

anisms illustrated in Fig. 2 generalize to the superposition of arbitrary many measures $\mu(\nu)$, $\nu = 1,\ldots,M$, since only the point that changes is the range of ν in $\alpha(\xi) = \min_\nu \alpha_\nu \xi)$. The most notable consequence is that *for $M \to \infty$ one almost surely observes the continuous phase transition* with the stopping point in $f(\alpha)$, since it suffices that one of the slopes of $\alpha_\nu(\xi)$ has a sign different from the others.

In Ref. [15] we extended our results to superpositions on two-scale Cantor sets by the use of the generalized thermodynamic formalism [4,16]. The picture developed in Fig. 2 is valid also in this case, since $f(\xi)$ is also concave, and $\alpha_\nu(\xi)$, now a nonlinear function, is still monotonic [5,15]. Although the expression for the critical value $q_{2\text{nd}}$ is more complicated, the condition for its existence is easily deduced from the sign of the slope of $\alpha_\nu(\xi)$ given by $\mathrm{sgn}[\ln p_0(\nu)/\ln l_0 - \ln p_1(\nu)/\ln l_1]$, where l_0 and l_1 are the scales of the Cantor set.

(b) Multinomial measures.—Multifractals organized on a p-nary tree with $P \geq 3$ lead to a multidimensional order parameter space. The partition function of the simplest example in this class, the superposition of M multiplicative measures $\mu(\nu)$ supported on a self-similar fractal set with $P = 3$ equal scales $1 \leq 1/P$, is given by

$$\begin{aligned} Z_N(q) &= \sum_{j=0}^{N}\sum_{k=0}^{N-j} T_N(j,k)P_N(j,k)^q\,, \\ P_N(j,k) &= \sum_{\nu=1}^{M} \pi(\nu)p_0(\nu)^j p_1(\nu)^k p_2(\nu)^{N-j-k}, \end{aligned} \tag{3}$$

with trinomial coefficients $T_N(j,k)$ and box contents $P_N(j,k)$. By the same reasoning as that for Eq. (2), one finds that the "free energy" $\tau(q)$ is determined by the minimum of a function $g(\xi_0;\xi_1;q)$, now of two arguments $\xi_0 = j/N$ and $\xi_1 = k/N$, given by $g(\xi_0,\xi_1;q) \equiv q\min_\nu \alpha_\nu(\xi_0,\xi_1) - s_T(\xi_0,\xi_1)$ with the three component entropy $s_T(\xi_0,\xi_1) = -\xi_0\ln\xi_0 - \xi_1\ln\xi_1 - (1-\xi_0-\xi_1)\ln(1-\xi_0-\xi_1)$ and $\alpha_\nu(\xi_0,\xi_1) = -\xi_0\ln p_0(\nu) - \xi_1\ln p_1(\nu) - (1-\xi_0-\xi_1)\ln p_2(\nu)$. Now, however, there exist more possible scenarios due to the possible shapes of the piecewise linear function $a(\xi_0,\xi_1) = \min_\nu a_\nu(\xi_0,\xi_1)$. For example, for $M = 2$ it may exhibit a *ridge* $\mathcal{L}$ in ξ space where $a_1(\xi_0,\xi_1) = a_2(\xi_0,\xi_1)$ as indicated in Fig. 3(b). The vector $\xi(q) \in \mathbb{R}^2$, which minimizes $g(\xi_0,\xi_1;q)$, eventually gets locked in a continuous way to the line $\mathcal{L}$ as q decreases from 0 to large negative values. This implies a second order transition to a new phase, which in this case is *not* a quenched phase, since $\xi(q)$ is still sensitive to external fields in the direction of $\mathcal{L}$. In addition, $\xi(q)$ reaches the boundary of the support of $s_T(\xi_0,\xi_1)$ [Fig. 3(a)] only for $q = -\infty$ since s_T vanishes there with an infinite slope. Correspondingly $\tau(q)$ is *not* linear in this new phase, and there is *no* stopping point in $f(\alpha)$. The later is obtained again as soon as the surface $a = a(\xi_0,\xi_1)$ exhibits a maximum at some

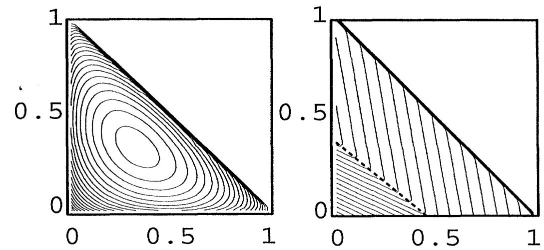

FIG. 3. Contour plots of (a) the fractal dimension $f(\xi_0,\xi_1) = s_T(\xi_0,\xi_1)/|\ln l|$ and (b) the singularity exponent $\alpha(\xi_0,\xi_1) = \min_\nu a_\nu(\xi_0,\xi_1)/|\ln l|$ for the superposition of $M = 2$ ternary Cantor measures. These functions generalize the picture developed in Fig. 2(b). The ridge in $\alpha(\xi_0,\xi_1)$ (dashed line) leads to a continuous transition with no stopping point in $f(\alpha)$. A stopping point is possible only for $M \geq 3$.

point ξ_c inside the support of $s_T(\xi_0,\xi_1)$. This becomes possible for $M \geq 3$, and is easily seen to be the generic case for large M. Note that by lowering q such a critical point ξ_c is now typically reached along a ridge $\mathcal{L}$. This implies the previous occurrence of an additional second order transition by a mechanism as for $M = 2$, with the consequence that generically the stopping point in $f(\alpha)$ does *not* coincide with an intersection point of the "pure" $f_\nu(\alpha)$ curves. As in case (a) the gross picture prevails if different length scales are allowed. For more components ($P > 3, \xi \in \mathbb{R}^{P-1}$) the appearance of a maximum in $a(\xi)$ and the corresponding quenched phase follows by the same reasoning. In addition, one expects that by lowering q multiple second order transitions occur by successive lockings into hyperplanes with decreasing dimension in ξ space.

(c) Markovian measures.—In the above examples the mass distribution ratio at some hierarchal level was independent of the masses attached in the previous level. Taking into account such a dependency results in the superposition of M Markovian measures $\mu(\nu)$, e.g., on a Cantor set as in (a). Each is characterized by a 2×2 transition matrix with elements $p_{ij}(\nu)$, $i, j \in \{0, 1\}$, obeying $\sum_j p_{ij} = 1$. It turns out that the partition function has exactly the form as in Eq. (3) with the following substitutions: $p_0 \to p_{00}$, $p_1 \to p_{11}$, $p_2 \to (p_{01}p_{10})^{1/2}$. Correspondingly the integers j and k count the pairs $0-0$ and $1-1$, respectively, in a binary address of length N of a fractal element (periodic boundary conditions for such a string are assumed). The coefficient $T_N(j,k)$ of Eq. (3) is replaced by the combinatorial factor $M_N(j,k)$, the number of possibilities to form such strings. This factor is known from the combinatorial solution of the $1-d$ Ising model [17] and leads to the entropy function $s_M(\xi_0,\xi_1) \equiv \lim_{N\to\infty} N^{-1}\ln M_N(\xi_0 N,\xi_1 N) = s_T(\xi_0,\xi_1) + (1-\xi_0+\xi_1)\ln(1-\xi_0+\xi_1) + (1+\xi_0-\xi_1)\ln(1+\xi_0-\xi_1) - (\xi_0+\xi_1)\ln 2$. It is easily seen that due to the terms in addition to $s_T(\xi_0,\xi_1)$ a

stopping point in $f(\alpha)$ becomes possible already for $M = 2$. Otherwise the discussion of case (b) holds here also.

To complete the picture we state our results in the language of statistical mechanics of Ising models. The function $s_M(\xi_0, \xi_1)$ is recognized as the entropy per spin in the microcanonical ensemble expressed in terms of $\xi_0 = N_{--}/N$ and $\xi_1 = N_{++}/N$, the number of "spin down" and "spin up" pairs, respectively [17]. Transforming to variables energy and magnetization $e = N^{-1}\sum_i S_i S_{i+1} = 2\xi_0 + 2\xi_1 - 1$ and $m = N^{-1}\sum_i S_i = \xi_1 - \xi_0$, $S_i = \pm 1$, one finds that $a_\nu(\xi_0, \xi_1) = [-J(\nu)\sum_i S_i S_{i+1} - h(\nu)\sum_i S_i - \bar{E}(\nu)]/N \equiv H_\nu(\{S_i\})/N$. The coupling constants $J(\nu)$ and the fields $h(\nu)$ are for each pure phase ν given by $J(\nu) = \frac{1}{4}\ln[p_{00}(\nu)p_{11}(\nu)/p_{01}(\nu)p_{10}(\nu)]$ and $h(\nu) = \frac{1}{2}\ln[p_{11}(\nu)/p_{00}(\nu)]$. The constant $\bar{E}(\nu) = (N/4)\ln[p_{00}(\nu)p_{01}(\nu)p_{11}(\nu)p_{10}(\nu)]$ is equal to $F_\nu(q = 1)$, the thermodynamic free energy of the Ising model at inverse temperature $q = 1$ [17]. It guarantees that for the pure phases $\tau_\nu(q)|\ln l| = q[F_\nu(q) - F_\nu(1)]/N$ vanishes at $q = 1$. The Hamiltonian corresponding to the superimposed Markovian multifractal measures is thus given by $H(\{S_i\}) = \min_\nu H_\nu(\{S_i\})$. The results of class (a) are obtained for the special case $p_{00}(\nu) = p_{10}(\nu) = p_0(\nu)$, $p_{11}(\nu) = p_{01}(\nu) = p_1(\nu)$, corresponding to $J(\nu) = 0$, with the identification $\xi = (1 - \xi_0 + \xi_1)/2 = (1 + m)/2$. The reduction to a one-dimensional problem occurs here because $a(\xi_0, \xi_1)$ depends only on the difference $\xi_0 - \xi_1$, implying that $\xi(q)$ is constrained to move on the parabola $e = m^2$. For the second limiting case $h(\nu) = 0$, i.e., $p_{00}(\nu) = p_{11}(\nu) \equiv p_0(\nu)$ and $p_{10}(\nu) = p_{01}(\nu) \equiv p_1(\nu)$, $a(\xi_0, \xi_1)$ is a function of $\xi_0 + \xi_1$ and $\xi(q)$ is constrained to lie on the bisectrix $\xi_0 = \xi_1$ implying $m = 0$. The constrained free energy functional $g(\xi_0 = \xi/2, \xi_1 = \xi/2; q)$ again turns out to be exactly the function $g(\xi; q)$ of case (a). Interestingly, for $h(\nu) = 0$ the second order transition exists only if one of the couplings is ferromagnetic, while the other is antiferromagnetic.

The fact that the Hamiltonians contain long-ranged multispin interactions is explicitly seen, e.g., for $M = 2$, by writing $H(\{S_i\})$ in the form $[H_1 + H_2 - (H_1 - H_2)\mathrm{sgn}(H_1 - H_2)]/2$, and by expanding the sgn function into a power series in the $\{S_i\}$. As a consequence of the long-ranged couplings a mean field theory leads exactly to the free energy "functional" $g(\xi; q)$ treated in (a) as is easily checked for $h(\nu) = 0$ or $J(\nu) = 0$. The critical exponents for the transition into the quenched phase, however, deviate from the usual mean field values. For all the examples treated in this paper the specific heat $-q^2\tau''(q)$ and the susceptibility jump from a finite value for $q > q_{2\mathrm{nd}}$ to zero in the quenched phase. This implies $\alpha = \gamma = 0$. The order parameter $\xi(q) - \xi_c$ increases linearly for $q > q_{2\mathrm{nd}}$ implying $\beta = 1$. We see that for this new universality class of models Rushbrooke's scaling law $\alpha + 2\beta + \gamma = 2$ is fulfilled in a trivial manner.

In conclusion, we have found the most probable elementary mechanisms for second order phase transitions in the thermodynamics of multifractals (for alternatives see [18]). Since superpositions of multifractal measures arise very naturally, our results appear to be relevant and observable in many experimental situations. They provide an alternative explanation for the common difficulty to measure full $f(\alpha)$ curves. We have restricted our presentation to analytically solvable classes. These, however, belong to the building blocks of general multifractal measures [19], and one can therefore expect a broad applicability of our findings.

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[1] See, e.g., H.G. Schuster, *Deterministic Chaos* (VCH, Weinheim, 1989).
[2] B.B. Mandelbrot, *The Fractal Geometry of Nature* (W.H. Freeman, New York, 1982).
[3] J. Feder, *Fractals* (Plenum, New York, 1988).
[4] T. Tél, Z. Naturforsch. A **43**, 1154 (1988).
[5] T.C. Halsey *et al.*, Phys. Rev. A **33**, 1141 (1986).
[6] C. Godrèche and J.M. Luck, J. Phys. A **23**, 3769 (1990); F. Axel and H. Terauchi, Phys. Rev. Lett. **66**, 2223 (1991).
[7] R. Bruinsma and G. Aeppli, Phys. Rev. Lett. **50**, 1494 (1983); S.N. Evangelou, J. Phys. C **20**, L511 (1987).
[8] R. Artuso, G. Casati, and D.L. Shepelyanski, Phys. Rev. Lett. **68**, 3826 (1992); R. Ketzmerick, G. Petschel, and T. Geisel, Phys. Rev. Lett. **69**, 695 (1992); I. Guarneri and G. Mantica, Phys. Rev. Lett. **73**, 3379 (1994).
[9] D. Katzen and I. Procaccia, Phys. Rev. Lett. **58**, 1169 (1987), and references therein.
[10] G. Radons, J. Stat. Phys. **72**, 227 (1993); for special aspects of dynamically generated multifractals see M. Ding *et al.*, Phys. Rev. Lett. **70**, 3872 (1993).
[11] H.G.E. Hentschel and I. Procaccia, Physica (Amsterdam) **8D**, 435 (1983).
[12] E. Ott, C. Grebogi, and J.A. Yorke, Phys. Lett. A **135**, 343 (1989).
[13] This has to be distinguished from cases where $f(\alpha_{\max}) > 0$ is reached with an infinite slope ($q = -\infty$) [4,5]. These are not associated with phase transitions.
[14] In the α range between the tangencies of $\alpha(\xi)$ with $f(\xi)$ one must set $f(\alpha) = \alpha$ (first order transition).
[15] G. Radons and R. Stoop, J. Stat. Phys. (to be published).
[16] M. Kohmoto, Phys. Rev. A **37**, 1345 (1988); R. Stoop, in *Encounter with Chaos,* edited by J. Peinke, J. Parisi, O.E. Roessler, and R. Stoop (Springer, Berlin, 1992).
[17] R.K. Pathria, *Statistical Mechanics,* (Pergamon Press, Oxford, 1972).
[18] G. Radons, Z. Naturforsch. A **49**, 1219 (1994).
[19] C.R. Handy and G. Mantica, Physica (Amsterdam) **43D**, 17 (1990).