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On the Formation of Black Holes in Nonsymmetric Gravity

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It has been recently suggested that the nonsymmetric gravitational theory (NGT) is free of black holes. Here we study the linear version of NGT. We find that even with spherical symmetry the skew part of the metric is generally nonstatic. In addition, if the skew field is initially regular, it will remain regular everywhere and, in particular, at the horizon. Therefore, in the fully nonlinear theory, if the initial skew field is sufficiently small, the formation of a black hole is to be anticipated.

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General relativity (GR) is a theory of gravitation, in which gravity is manifested by the curvature of spacetime, which is described by Riemannian geometry [1]. Field theories which use non-Riemannian geometry have been formulated by Einstein [2], Schrödinger [3], Einstein and Straus [4], and more recently by Moffat [5,6] and Klotz [7]. In non-Riemannian geometry the metric tensor $g_{\mu\nu}$ is not assumed to be symmetrical in its two indices. This property complicates the geometry considerably, and induces torsion in spacetime. Einstein [2] formulated a nonsymmetric field theory as part of his quest for a unified field theory, namely, a unified theory of classical gravity and electromagnetism.

Recently, there has been growing interest in nonsymmetric gravitation theory (NGT) for motivations different than Einstein's. Cornish and Moffat (CM) [8,9] studied a class of exact static spherically symmetric solutions to the NGT field equations. This class depends on the two parameters m and s, where m is the source's mass, and s determines the strength of the skew part of the metric tensor (that is, at large distance, $r \gg m$, this skew part is proportional to s). In all these solutions, there are no trapped surfaces, and consequently there are no black holes. Based on these static solutions, CM suggested that NGT was free of black holes (and, thereof, of spacetime singularities) [8–10].

It is remarkable that even for arbitrarily small s the static skew field "destroys" the horizon. That is, even if the skew field (i.e., the skew part of the metric tensor) is arbitrarily small at $r \gg m$, in the static solution it grows in an uncontrolled way on the approach to $r = 2m$—until it becomes so strong that it modifies the geometry dramatically and prevents the formation of trapped surfaces. This behavior is nicely demonstrated in the context of linearized NGT. In linearized NGT, the skew field is regarded as an infinitesimally small perturbation over the standard, symmetric metric. The linear analog of the model analyzed by CM is that of a static, spherically symmetric, linearized skew field on a Schwarzschild background. One then finds (see below) that the linearized skew field diverges at $r = 2m$. This linear divergence indicates the effectiveness of the skew field, and its ability to "destroy" the black hole, in the context of fully nonlinear NGT.

Of course, before making any definitive statements about the existence or nonexistence of black holes in NGT, one must address the following question: Is the above mentioned phenomenon (the absence of a black hole in the static spherically symmetric solutions) a generic characteristic of NGT or a result of the symmetry (staticity) imposed? Answering this question requires an investigation of the nature of the generic dynamical NGT solutions. This is an extremely hard task, because NGT is much more complicated than GR (and, of course, its general dynamic solution is as yet unknown). Fortunately, it is possible to translate the above question to the context of linearized NGT: Is the divergence of the linearized skew field at the Schwarzschild radius a generic feature of linearized NGT or a consequence of the assumption of staticity (of the skew field)? If the generic linearized solution was divergent at the background's horizon, an important dynamical effect would be anticipated in the fully nonlinear NGT. On the other hand, if the linearized skew field is found to be generically regular at the horizon, the situation is different: Then (at least for a sufficiently small initial skew field) the linearized solution is likely to be a good approximation to the full theory, and no drastic effects are expected to occur at the horizon. In such a case, we should expect a gravitational collapse to proceed pretty much like in GR—and, in particular, a formation of a black hole is to be anticipated.

The goal of this Letter is to address the above question. We shall study the dynamical behavior of a linearized skew field on a GR background and apply this formalism to a spherically symmetric skew field on a Schwarzschild background. We shall show that the linearized equation possesses a well-posed initial-value formulation. An important result is that, even in spherical symmetry, the skew field need not be static. Moreover, for regular initial data on some spacelike hypersurface Σ, no divergence occurs anywhere in the entire domain of dependence. In particular, the dynamics of the skew field at the horizon is perfectly regular. Our conclusion is, therefore, that if the initial skew field is sufficiently small, a black hole is likely to form in gravitational collapse—just like in standard GR.

The vacuum field equations of NGT are

$$g_{\mu\nu,\sigma} - g_{\rho\nu}\Gamma^{\rho}_{\mu\sigma} - g_{\mu\rho}\Gamma^{\rho}_{\sigma\nu} = 0\,, \tag{1}$$

$$\left(\sqrt{-g}\,g^{[\mu\nu]}\right)_{,\nu} = 0\,, \tag{2}$$

$$R_{(\alpha\beta)} = 0\,, \tag{3}$$

$$R_{[\alpha\beta],\gamma} + R_{[\beta\gamma],\alpha} + R_{[\gamma\alpha],\beta} = 0\,, \tag{4}$$

where $g_{\mu\nu}$ is the nonsymmetric metric tensor, g is its determinant, $R_{\alpha\beta}$ is the generalized Ricci tensor [see Eq. (8) below], and $\Gamma^{\alpha}_{\beta\gamma}$ is the nonsymmetric affine connection. The inverse metric $g^{\mu\nu}$ is defined by $g^{\mu\nu}g_{\mu\sigma} = g^{\nu\mu}g_{\sigma\mu} = \delta^{\nu}_{\sigma}$.

We now consider the linearized NGT. Namely, we assume that the skew part of the metric tensor, $h_{\mu\nu}$, is a small perturbation over the symmetric GR metric, and develop the field equation to first order in this perturbation. Denoting all background fields by a caret, we write $g_{\mu\nu} \equiv \hat{g}_{\mu\nu} + h_{\mu\nu}$, $\Gamma^{\rho}_{\mu\nu} \equiv \hat{\Gamma}^{\rho}_{\mu\nu} + D^{\rho}_{\mu\nu}$, and $R_{\alpha\beta} \equiv \hat{R}_{\alpha\beta} + Q_{\alpha\beta}$. Here $\hat{g}_{\mu\nu}$ is a standard, symmetric GR metric and $\hat{\Gamma}^{\rho}_{\mu\nu}$ and $\hat{R}_{\alpha\beta}$ are the standard connection and Ricci tensor, respectively, associated with this background metric. Note the symmetry features of the various entities: By definition, we have $\hat{g}_{(\mu\nu)} = \hat{g}_{\mu\nu}$, $\hat{\Gamma}^{\rho}_{(\mu\nu)} = \hat{\Gamma}^{\rho}_{\mu\nu}$, $\hat{R}_{(\alpha\beta)} = \hat{R}_{\alpha\beta}$, and $h_{[\mu\nu]} = h_{\mu\nu}$. We shall show below that $D^{\alpha}_{[\beta\gamma]} = D^{\alpha}_{\beta\gamma}$ and $Q_{[\alpha\beta]} = Q_{\alpha\beta}$. The background metric $\hat{g}_{\mu\nu}$ is taken to be vacuum, i.e., $\hat{R}_{\mu\nu} = 0$.

From the metricity equation (1) we find, to the linear order in the skew field, that

$$\hat{g}_{\rho\beta}D^{\rho}_{\alpha\gamma} + \hat{g}_{\alpha\rho}D^{\rho}_{\gamma\beta} = h_{\alpha\beta;\gamma}\,, \tag{5}$$

where a semicolon denotes covariant differentiation with respect to the GR background $\hat{g}_{\mu\nu}$. Solving Eq. (5) we find that

$$D^{\alpha}_{\beta\gamma} = D^{\alpha}_{[\beta\gamma]} = \frac{1}{2}\,\hat{g}^{\alpha\delta}(h_{\delta\gamma;\beta} + h_{\beta\delta;\gamma} + h_{\beta\gamma;\delta})\,. \tag{6}$$

Next we linearize Eq. (2). To linear order we have $g^{[\mu\nu]} = -h^{\mu\nu}$. (We use the background metric $\hat{g}_{\alpha\beta}$ to raise or lower indices.) Equation (2) is thus reduced to

$$h^{\alpha\beta}_{;\beta} = h^{\beta\alpha}_{;\beta} = 0\,. \tag{7}$$

The generalized (Hermitianized) Ricci tensor is defined in NGT by [2]

$$R_{\alpha\beta} = \Gamma^{\rho}_{\alpha\beta,\rho} - \frac{1}{2}\left(\Gamma^{\rho}_{(\alpha\rho),\beta} + \Gamma^{\rho}_{(\rho\beta),\alpha}\right) - \Gamma^{\rho}_{\alpha\sigma}\Gamma^{\sigma}_{\rho\beta} + \Gamma^{\rho}_{\alpha\beta}\Gamma^{\sigma}_{(\rho\sigma)}\,. \tag{8}$$

Expanding this equation to the first order in the perturbation, we find that

$$Q_{\alpha\beta} = Q_{[\alpha\beta]} = D^{\rho}_{\alpha\beta;\rho}\,, \tag{9}$$

or, equivalently,

$$Q_{\beta\gamma} = \frac{1}{2}\,\hat{g}^{\alpha\delta}(h_{\delta\gamma;\beta\alpha} + h_{\beta\delta;\gamma\alpha} + h_{\beta\gamma;\delta\alpha})\,. \tag{10}$$

Recalling the noncommutivity of covariant derivatives, we rewrite Eq. (10) as

$$Q_{\beta\gamma} = \frac{1}{2}\,\hat{g}^{\delta\alpha}(h_{\delta\rho}\hat{R}^{\rho}{}_{\gamma\beta\alpha} + h_{\rho\gamma}\hat{R}^{\rho}_{\delta\beta\alpha} + h_{\beta\rho}\hat{R}^{\rho}{}_{\delta\gamma\alpha} + h_{\rho\delta}\hat{R}^{\rho}{}_{\beta\gamma\alpha}) + \frac{1}{2}\left(h^{\alpha}{}_{\gamma;\alpha\beta} + h_{\beta}{}^{\alpha}{}_{;\alpha\gamma} + \hat{g}^{\delta\alpha}h_{\beta\gamma;\delta\alpha}\right), \tag{11}$$

where $\hat{R}^{\rho}_{\gamma\beta\alpha}$ is the background Riemann curvature tensor. In view of $\hat{R}_{\alpha\beta} = 0$ and Eq. (7), Eq. (11) becomes

$$Q_{\beta\gamma} = \frac{1}{2}\,\hat{g}^{\delta\alpha}h_{\beta\gamma;\delta\alpha} + 2\hat{g}^{\delta\alpha}h_{\delta\rho}\hat{R}^{\rho}{}_{\alpha\beta\gamma}\,. \tag{12}$$

Linearizing Eqs. (3) and (4), one finds that the former is automatically satisfied by $Q_{\beta\gamma}$, and Eq. (4) reduces to

$$Q_{\alpha\beta,\gamma} + Q_{\gamma\alpha,\beta} + Q_{\beta\gamma,\alpha} = 0\,. \tag{13}$$

Equation (13) [with the identity (12)] together with the constraint (7) are the linearized vacuum NGT equations for $h_{\mu\nu}$.

Our analysis so far was quite generic: We did not make any assumptions about any symmetry of either $\hat{g}_{\mu\nu}$ or $h_{\mu\nu}$. We shall now restrict our attention to spherical symmetry. Namely, we shall take $\hat{g}_{\mu\nu}$ to be Schwarzschild, and $h_{\mu\nu}$ to be spherically symmetric. We start from the spherically symmetric metric used by CM [see, e.g., Eq. (5) in Ref. [8]], and allow the three nontrivial metric functions—namely, α, γ, and f—to depend on both r and t:

$$g_{\mu\nu} = \begin{pmatrix} \gamma(r,t) & 0 & 0 & 0 \\ 0 & -\alpha(r,t) & 0 & 0 \\ 0 & 0 & -r^2 & f(r,t)\sin\theta \\ 0 & 0 & -f(r,t)\sin\theta & -r^2\sin^2\theta \end{pmatrix}. \tag{14}$$

(The most general spherically symmetric metric may also include a nonzero metric function $g_{[rt]}$ [11]. Here we follow CM and restrict our attention to the simpler case, $g_{[rt]} = 0$.) We now linearize the field equations in f. The zeroth-order equation $\hat{R}_{\mu\nu} = 0$ immediately implies that the background metric is the Schwarzschild solution: $\gamma = 1/\alpha = 1 - 2m/r$, so we only need to calculate f. Equation (7) is automatically satisfied by the skew part of (14), and we only need to consider Eq. (13). A straightforward calculation, based on Eq. (12), yields that the only nonvanishing components of $Q_{\mu\nu}$ are

$$Q_{\theta\phi} = -Q_{\phi\theta} = \left[\frac{1}{2}\left(\frac{\ddot{f}}{\gamma} - \frac{f''}{\alpha}\right) + \frac{f'}{\alpha r} + \frac{1}{2}\frac{f'\alpha'}{\alpha^2} - 2\frac{f\alpha'}{\alpha^2 r}\right]\sin\theta, \tag{15}$$

where a dot and a prime denote partial differentiation with respect to t and r, correspondingly. [We have also derived this equation directly, by calculating $R_{\mu\nu}$ from the (time-dependent) metric (14) in the fully nonlinear NGT, and then linearizing it in f.] From Eq. (13) it is obvious that $Q_{\theta\phi}$ cannot depend on r or t. The most general solution of this equation is, therefore, $Q_{\theta\phi} = -c\sin\theta$, where c is some real constant (see also Ref. [12]). It can be shown, however, that for $c \neq 0$ the spacetime is not asymptotically Minkowski [13]. We shall therefore focus our attention here on the case $c = 0$. The field equation for f will thus be

$$\frac{1}{2}\left(\frac{\ddot{f}}{\gamma} - \frac{f''}{\alpha}\right) + \frac{f'}{\alpha r} + \frac{1}{2}\frac{f'\alpha'}{\alpha^2} - 2\frac{f\alpha'}{\alpha^2 r} = 0. \tag{16}$$

In the static limit, i.e., when $\dot{f}$ is taken to vanish, we recover from Eq. (16) the linear analog of the CM equation for f [see, in particular, Eq. (2.4) of Ref. [14]]. One can easily verify that, in the static limit, the linearized f diverges logarithmically at $r = 2m$. This is just the linear analog of the behavior found by CM. Here, however, we are in a position to study the dynamical content of the theory.

Equation (16) is a linear, second-order, hyperbolic, partial differential equation, and, consequently, it possesses a well-posed initial-value formulation. Thus, given f and $\dot{f}$ on some spacelike surface, standard theorems guarantee the existence and uniqueness of a regular solution $f(r,t)$ throughout the domain of dependence (or, more precisely, as long as the background metric tensor is regular). This, by itself, proves that f does not satisfy a generalized Birkhoff's theorem [15]. Namely, despite the spherical symmetry, f is generically dynamic (for one is allowed to choose nonzero initial $\dot{f}$).

The next stage of our analysis is to study the behavior of f at the horizon. The Schwarzschild coordinates are unsuitable for that purpose as they go singular at $r = 2m$. We therefore need to transform to some other spherical coordinates (e.g., Kruskal and Szekeres [16]). This transformation is most easily done by expressing Eq. (16) in a covariant form. Defining a new function $k(r,t) \equiv f(r,t)/r^2$, one readily finds that Eq. (16) reduces to

$$\hat{g}^{\mu\nu}k_{;\mu\nu} + \frac{2}{r^2}k = 0. \tag{17}$$

Take now any coordinates that cover the Schwarzschild manifold (such as Kruskal and Szekeres) and reexpress Eq. (17) in terms of partial derivatives. The resultant equation is obviously a linear, second-order, hyperbolic, partial differential equation—throughout the spacetime (with coefficients which are regular everywhere). Therefore, for any partial Cauchy surface Σ in the analytically extended Schwarzschild spacetime, and for any choice of regular k (or f) and its time derivative on it, the existence and uniqueness of a regular solution $k(r,t)$ [or $f(r,t)$] throughout $D^+(\Sigma)$ is guaranteed. In particular, f is regular at the horizon.

We have found that if the linearized skew function f is initially regular, it will remain regular throughout the domain of dependence (except possibly at $r = 0$) and, in particular, at the event horizon. Note that there is no conflict between this result and the divergence of the static linearized skew field at $r = 2m$. From the initial-value point of view, the linearized static solution fails to be regular at $r = 2m$ simply because it evolved from singular initial data. (That is, in view of the staticity, the divergence at $r = 2m$ must have existed already on the initial slice.) For any regular initial data, however, the skew field will remain regular at the horizon.

Let us now discuss the implication of the above results to nonlinear NGT. Generally, one expects a linear perturbation analysis to be a good approximation to the original nonlinear theory as long as the perturbation is small. If, however, the linearized perturbation develops a divergence at some point, this may break the validity of the linear approximation. Indeed, the divergence of the static linearized skew field at the horizon indicates strong nonlinear effects, which completely modify the GR geometry (at $r \leq 2m$). We have found, however, that if the initial data for the linear case are regular, no divergence will occur. We therefore arrive at the following conclusion regarding the behavior of the fully nonlinear system: If the skew function f and its time derivative are regular and sufficiently small at the initial moment, they are likely to remain small, and dynamically unimportant, in the neighborhood of $r = 2m$. In particular, a black hole is expected to form—pretty much like in GR. (Important dynamical effects are possible, however, near $r = 0$.) Again, there is no conflict between this result and the strong nonlinear effect found by CM in the static case, because in the latter the initial skew field is necessarily strong near $r = 2m$.

Strictly speaking, the above considerations are restricted to the vacuum case, i.e., to the analytically extended Schwarzschild spacetime. One may therefore be concerned about the validity of our conclusion to the situation of gravitational collapse (in spherically symmetric gravitational collapse matter must always be involved). The present authors regard this as a technical difficulty rather than an inherent one. Although our regularity arguments are not strictly valid in the presence of matter, in view of the above analysis there is no positive indication whatsoever for any anomalous behavior of the skew field at the horizon (given regular and sufficiently small initial data).

In addition, let us imagine a nonspherical GR background $\hat{g}_{\alpha\beta}$ describing a dynamical gravitational collapse of pure gravitational radiation (which in GR produces a black hole [17]). Consider now a small (linearized) skew perturbation $h_{\alpha\beta}$ over this background. (We assume that the initial data for the skew field are given on an initial hypersurface prior to the formation of the black hole.) The vacuum field equations are certainly valid in that case. Although our above initial-value analysis is restricted to spherical symmetry, it is possible to extend it to the generic (nonspherical) case [18]. This general analysis is beyond the scope of the present Letter, so we shall just outline it briefly. In the generic case, one can introduce a "vector-potential" A_μ (A_μ is closely related to the vector W_μ of Ref. [5]), such that $Q_{\mu\nu} = A_{[\mu,\nu]}$. [This automatically solves Eq. (13).] Using the Lorentz gauge, $A^\mu{}_{;\mu} = 0$, one can derive a system of second-order linear hyperbolic differential equations for A_μ and $h_{\mu\nu}$, which is consistent with the constraint equations [i.e., with Eq. (7) and $A^\mu{}_{;\mu} = 0$]. Doing so, we again obtain a well-posed initial-value formulation for the generic evolution of the linearized nonsymmetric field. One can now repeat the above arguments and arrive at a similar conclusion—this time, applied to the formation of a nonspherical black hole by the collapse of pure gravitational radiation: If the initial skew field is sufficiently small, no important dynamical effects are expected to occur on the approach to the event (or apparent) horizon. Therefore, a black hole is expected to form, as in GR.

If, indeed, a black hole forms in NGT, what would then be its final state? The equation satisfied by k [Eq. (17)] is nothing but the radial equation for the $l = 1$ mode of a massless scalar field. Consequently, from the analysis of Price [19], an external observer will witness an inverse power-law decay (in the external time t) of the skew field, with a usual GR black hole as the final state. [It is interesting to note that in the more general case, $c \neq 0$, a permanent skew hair will remain after perturbations die out: Defining $\tilde{k} \equiv k + c$, one readily finds that Eq. (17) is recovered with k replaced by $\tilde{k}$. This means that $\tilde{k}$ will decay asymptotically to zero, and consequently k will approach $-c$. The nature of this hair, and the implications it has on the features of the black hole, await further investigation. Recall, however, that this case is not asymptotically Minkowski.]

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