Scaling Relations for a Randomly Advected Passive Scalar Field

Robert H. Kraichnan
369 Montezuma 108, Santa Fe, New Mexico 87501-2626

Victor Yakhot
Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544

Shiyi Chen
IBM T. J. Watson Research Center, Yorktown Heights, New York 10598
and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
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A recent ansatz for dissipation terms gave anomalous inertial-range scaling exponents ($\propto n^{1/2}, n \to \infty$) for the nth-order structure functions of a passive scalar field advected by a random velocity field. Analysis of a series expansion for the conditional mean of a dissipation term suggests that the ansatz gives the only possible anomalous scaling. Anomaly of inertial-range scaling is supported by realizability inequalities on the dissipation field. Predictions for conditional means and structure functions are compared with simulations.

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The limit of infinitely rapid change in time of a random velocity field $\mathbf{u}(\mathbf{x},t)$ that advects a scalar field is of interest because many results can be obtained exactly while, nevertheless, the one-particle and two-particle diffusivities can be similar to those of more realistic velocity fields.

A recent Letter [1] considered the structure functions $S_{2n}(r) = \langle |\Delta T(\mathbf{r})|^{2n}\rangle$ of a passive scalar field $T(\mathbf{x},t)$ that obeys

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x},t) \cdot \boldsymbol{\nabla}\right)T(\mathbf{x},t) = \kappa \nabla^2 T(\mathbf{x},t). \quad (1)$$

Here $\Delta T(\mathbf{r})$ denotes $T(\mathbf{x}+\mathbf{r}) - T(\mathbf{x})$, $\langle \cdots \rangle$ denotes ensemble average over homogeneous, isotropic statistics, and κ is molecular diffusivity. In the rapid-change limit, the exact evolution equation for $S_{2n}(r)$ is

$$\frac{\partial S_{2n}(r)}{\partial t} - \frac{2}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\eta(r)\frac{\partial S_{2n}(r)}{\partial r}\right) = \kappa J_{2n}(r), \quad (2)$$

where d is space dimensionality, $\eta(r)$ is the two-particle eddy-diffusivity scalar defined by

$$\eta(r) = \frac{1}{2}\int_0^t \langle [\delta_{\|}u(\mathbf{r},t)\delta_{\|}u(\mathbf{r},t')\rangle dt', \quad (3)$$

with $\delta_{\|}u(\mathbf{r},t) = [\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{x}+\mathbf{r},t)]\cdot \mathbf{r}/r$, and

$$J_{2n}(r) = 2n\langle [\Delta T(\mathbf{r})]^{2n-1}(\nabla_x^2 + \nabla_{x'}^2)\Delta T(\mathbf{r})\rangle. \quad (4)$$

The velocity field is switched on at $t = 0$. $T(\mathbf{x}, t = 0)$ is Gaussian.

Equation (4) may be written

$$J_{2n}(r) = 2n\langle [\Delta T(\mathbf{r})]^{2n-1} H[\Delta T(\mathbf{r})]\rangle, \quad (5)$$

where

$$H[\Delta T(\mathbf{r})] = \langle (\nabla_x^2 + \nabla_{x'}^2)\Delta T(\mathbf{r}) | \Delta T(\mathbf{r})\rangle \quad (6)$$

and $\langle \cdot | \Delta T(\mathbf{r})\rangle$ denotes the ensemble average conditioned on a given value $\Delta T(\mathbf{r})$.

Equations (2) for all n are implied by a single equation for $P(\Delta T, r, t)$, the one-point probability distribution function (PDF) of $\Delta T(\mathbf{r})$ [2,3]:

$$\frac{\partial P}{\partial t} - \frac{2}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\eta(r)\frac{\partial P}{\partial r}\right) + \kappa \frac{\partial}{\partial \Delta T}[H(\Delta T)P] = 0. \quad (7)$$

H is a function of ΔT whose form depends on r. It can be expanded formally as

$$H[\Delta T(\mathbf{r})] = f_1(r)\Delta T(\mathbf{r}) + f_3(r)[\Delta T(\mathbf{r})]^3 + \cdots, \quad (8)$$

whence

$$J_{2n}(r) = 2n[f_1(r)S_{2n}(r) + f_3(r)S_{2n+2}(r) + \cdots]. \quad (9)$$

Only odd powers appear in (8) because of the assumed symmetries. How can $H[\Delta T(\mathbf{r})]$ be a nonlinear function of $\Delta T(\mathbf{r})$ when (1) is linear in T? Both $T(\mathbf{x},t)$ and $\nabla^2 T(\mathbf{x},t)$ are linear functionals of the initial Gaussian T field. But they are different, nonlinear functionals of the advecting velocity field u. Therefore the statistical relation between them also is nonlinear. The $f_j(r)$ are averages over T and $\mathbf{u}$ of such form that (8) is homogeneous of degree 1 in T.

If $\kappa = 0$, only the f_1 term survives in (8) at $r \to \infty$. This result does not constrain the velocity statistics or require that the velocity field change rapidly. The crucial fact is that T is carried by the flow unchanged in value. $\nabla^2 T(\mathbf{x},t)$ is a nonlocal, linear functional of the initial

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Gaussian field. $T(\mathbf{x}, t)$ is a stochastic rearrangement of the initial field and has the same Gaussian distribution at any point $\mathbf{x}$. The statistical dependence between two Gaussian variables a and b has the linear form $a = cb + s$, where c is a correlation coefficient and s is a Gaussian variable independent of b. It follows from this that $\nabla^2 T(\mathbf{x}, t)$ has the form $c(\mathbf{x}, t)T(\mathbf{x}, t) + s(\mathbf{x}, t)$, where c and s are functionals of the velocity field but are independent of T. Hence $\langle \nabla^2 T | T \rangle = \bar{c}T$, where $\bar{c}$ is an average over all realizations of the homogeneous velocity field over the interval $(0, t)$. It follows that H is linear in $\Delta T(r)$ if r is large enough compared to any correlation length of the scalar field.

Consider the truncation of (8) to the $f_1(r)$ term at $\kappa > 0$. Multiplication of (8) by $\Delta T(\mathbf{r})$, averaging over ensemble, and the use of homogeneity then provide an immediate evaluation of $f_1(r)$ and $J_{2n}(r)$:

$$f_1(r) = A(r)/S_2(r), \tag{10}$$

$$J_{2n}(r) = 2nS_{2n}(r)A(r)/S_2(r), \tag{11}$$

where $A(r) = \nabla^2 S_2(r) - \nabla^2 S_2(r = 0)$. Equation (11) is the ansatz for $J_{2n}(r)$ proposed in [1]. It is exact for $n = 1$, and the $S_2(r)$ equation fixes $A(r)$. Equation (11) was invoked in [1] as an approximation that is valid over a range of statistics from Gaussian to strongly intermittent. Additional arguments for (11) will be developed here. Linearity approximations on conditional means like $H(\Delta T)$, and relations like (11), were studied earlier by Ching [4] and by Pope and Ching [5].

Assume that there is a scaling range in which $\eta(r) \propto r^{\zeta(\eta)}$ $[0 < \zeta(\eta) < 2]$, $S_{2n}(r) \propto r^{\zeta_{2n}}$, and $f_j(r) \propto r^{z_j}$. Then (10) and (11) imply $\zeta_2 = 2 - \zeta(\eta)$, $z_1 = -\zeta_2$, and [1]

$$\zeta_{2n} = \tfrac{1}{2}\sqrt{4nd\zeta_2 + (d - \zeta_2)^2} - \tfrac{1}{2}(d - \zeta_2), \tag{12}$$

whence $\zeta_{2n} \approx \sqrt{nd\zeta_2}$ $(n \gg 1)$ and $\zeta_{2n} \approx n\zeta_2$ $(d \gg 1)$.

Further analysis of (9) requires that two kinds of scaling be defined: regular scaling, in which $\zeta_{2n} = n\zeta_2$, and anomalous scaling, in which $\zeta_{2n+2} - \zeta_{2n}$ is a decreasing function of n [6]. Regular scaling, with $\zeta_{2n} = 2n$, holds in the far-dissipation range $r \to 0$ if the relation $\Delta T(\mathbf{r}) \approx \nabla T \cdot \mathbf{r}$ is valid for small enough r. The latter is not an *a priori* necessity, but if it does not hold, moments of the PDF of $|\nabla T|$ of order >2 do not exist.

Consider (2) in steady state over the entire range of r, with stationarity maintained by adding a rapidly changing macroscale source term $b(\mathbf{x}, t)$ to the right side of (1). The source induces the term $2n(2n - 1)S_{2n-2}B(r)$ on the right side of (2), where $B(r)$ is a structure function that measures source strength [1]. If (9) is taken, (2) for all n is a set of simultaneous linear equations for the semi-infinite set of functions $S_{2n}(r)$, with the single inhomogeneous term $2B(r)$ in the $n = 1$ equation. The coefficient functions are the semi-infinite set $f_j(r)$.

The $B(r)$ term is negligible in an inertial scaling range of r; there (2) and (9) yield (cf. [1])

$$\zeta_{2n}(\zeta_{2n} + d - \zeta_2) = nd\zeta_2[\bar{f}_1(r) + \bar{f}_3(r)c_{2n+2,2n}r^{\zeta_{2n+2}-\zeta_{2n}} + \bar{f}_5(r)c_{2n+4,2n}r^{\zeta_{2n+4}-\zeta_{2n}} + \cdots], \tag{13}$$

where $c_{m,n} \equiv S_m(1)/S_n(1)$ and $\bar{f}_j(r) = S_2(r)f_j(r)/A(r)$.

The left side of (13) is r independent. There are two simple ways for the right side to be also. One is $\bar{f}_j(r) = 0$ $(j > 1)$, which yields $\bar{f}_1(r) = 1$ at $n = 1$ and the anomalous scaling (12). The second is regular scaling with $\bar{f}_j(r) \propto r^{-(j-1)\zeta_2/2}$. This corresponds to the general scaling forms $H(\Delta T) = r^{-\zeta_2/2}h(\Delta T/r^{\zeta_2/2})$ and $P(\Delta T) = r^{-\zeta_2/2}p(\Delta T/r^{\zeta_2/2})$, where h and p are functions so far undetermined. The question of interest now is whether more general forms of anomalous scaling are consistent with (13).

The geometric series $g(r) = \sum_{n=0}^{\infty} c_n r^{n\alpha}$ with $\alpha > 0$ is independent of r only if all c_n vanish for $n > 0$. This is established formally by setting derivatives to zero at $r = 0$. A similar formal result follows for the more general series $g(r) = \sum_{n=0}^{\infty} c_n r^{\alpha_n}$, where the α_n are positive and increase monotonically, but nonlinearly, with n. Thus let $\lambda = r^{\alpha_1}$ and set $dg/d\lambda = 0$ $(r = 0)$ to get $c_1 = 0$, then let $\lambda = r^{\alpha_2}$ to get $c_2 = 0$, and so on. To apply this to (13), assume that each inertial-range function $\bar{f}_j(r)$ is dominated by a single power of r. Pick a particular (large) value of n and choose these powers to make each term on the right side of (13) independent of r. Then take various smaller values of n. For each such n, the right side of (13) is a series in ascending powers of r and can be independent of r only if all the $\bar{f}_j(r)c_{2n+m,2n}$ vanish for $j > 1$.

We do not claim a proof that (12) is the uniquely possible anomalous inertial-range scaling, because we have not ruled out exotic forms of H for which the expansions (8) and (13) have no meaning. Moreover, one cannot assert from first principles that there actually is power-law scaling of $S_{2n}(r)$ $(n > 1)$ in the inertial range.

Whatever goes on in the inertial range, regular scaling is expected in the far-dissipation range $r \to 0$. The ansatz (11) cannot hold in this range, because it does not support stationarity at large n. As $r \to 0$, $\eta(r) \propto r^2$. The left side of (2) goes like n^2 at large n, while the right side goes like n, if (11) holds. This can be remedied if the summed series (9) has the form of a prefactor multiplying an expression like $(1 + x)^{1/m}$, where x contains some appropriate power of n. If the PDF of $|\nabla T|$ is exponential in form, $x \propto n^2$, $m = 2$ is a possible resolution.

Consider an initial Gaussian state with $S_2(r)$ at steady-state value so that H is linear and (10) holds for all r. The evolution suggested by our analysis is that $H[\Delta T(r)]$ stays unchanged at $r \gg \ell_d$ (ℓ_d is a dissipation scale) but curves up above linearity for $r \lesssim \ell_d$. By (10) at

$r \to \infty$, a corollary would be $\langle \nabla^2 T | T \rangle \propto T$ at infinite Péclet number, which implies Gaussian one-point PDF of T. Some deviation from linearity of $H[\Delta T(r)]$ is expected in an inertial range of only finite extent.

Equation (11) implies scaling relations for dissipation fluctuations on inertial-range scales. A spatial partial integration on the left side leads to $\langle |\nabla T(\mathbf{x}') - \nabla T(\mathbf{x})|^2 | \Delta T \rangle \propto |\Delta T|^2 / S_2(r)$ at inertial range r. This can be shown to imply $\langle \chi(\mathbf{x} + \mathbf{r}) \chi(\mathbf{x}) \rangle \propto S_4(r)/[S_2(r)]^2$, where $\chi(\mathbf{x}) = |\nabla T(\mathbf{x})|^2$, or

$$\langle \chi(\mathbf{x} + \mathbf{r}) \chi(\mathbf{x}) \rangle \approx \bar{\chi}^2 (\ell_0 / r)^{2\zeta_2 - \zeta_4} \, (\ell_0 \gg r \gg \ell_d), \tag{14}$$

where $\bar{\chi} = \langle \chi(\mathbf{x}) \rangle$ and ℓ_0 is a macroscale (source). Equation (14) describes intermittency of dissipation that increases with Péclet number in a way suggestive of Kolmogorov's refined similarity hypothesis [7].

Can the formal regular inertial-range scaling represent a physical solution? Regular scaling means that intermittency does not increase cumulatively as cascade proceeds, and this requires that the intermittency increase built in each cascade step be fully relaxed by the spatial smoothing associated with eddy diffusivity.

A consequence plausibly expected from such total relaxation at each step is that the dissipation-range excitation is statistically independent of inertial-range excitation at infinite Péclet number [8]. This implies

$$J_{2n}(r) = 2n(2n - 1) S_{2n-2}(r) A(r) \tag{15}$$

at inertial range r, which, used in (2), generates all the $S_{2n}(r)$ by recursion $[S_0(r) \equiv 1]$. In particular, $S_4(r)/[S_2(r)]^2 = 3d/(d + \zeta_2)$ and $S_{2n}(r)/[S_2(r)]^n \approx (2d/\zeta_2)^n$ for large n. The result is regular scaling, but with a flatness less than 3 and high-order $S_{2n}(r)$ corresponding to a $P(\Delta T)$ with an absolute cutoff at a value of $|\Delta T|$ that is $O(\sqrt{S_2(r)})$. Neither result is a possible consequence of a Gaussian velocity field acting on scalar gradients.

Regular scaling therefore needs statistical dependence between dissipation and inertial scales. Let the inertial range of wave numbers be divided into bands of uniform logarithmic width. Each band represents a subfield in $\mathbf{x}$ space, which we model by a Gaussian field multiplied by a positive, $\mathbf{x}$-dependent modulation factor, whose normalized statistics are independent of band center wave number. Thus $S_4(r) \approx 3c[S_2(r)]^2$ in the inertial range, where $c > 1$ is r independent. Then (2) requires that $J_4(r)$ exceed the right side of (15) by a factor $\approx c(d + \zeta_2)/d$. If this is so for all $\ell_0 \gg r \gg \ell_d$, realizability inequalities require that $\langle \chi^2 \rangle / \bar{\chi}^2$ be at least $O(\ln(\ell_0/\ell_d))$ as $\ell_0/\ell_d \to \infty$: $\chi(\mathbf{x})$ must have spectral support, in each decade of the inertial range, that contributes $O(\bar{\chi}^2)$ to $\langle \chi^2 \rangle$.

Such behavior of the dissipation field is inconsistent with regular scaling, which requires that the flatness factors of the band-limited scalar fields do not increase with band wave number: There is no way in which strong intermittency of dissipation can be supported by cascade (stretching) unless there is similarly strong intermittency in the bands near the top of the inertial range. Regular scaling of $S_4(r)$ (and, similarly, higher S_{2n}) therefore seems impossible.

Perturbation analysis [9] that exploits the rapid change of $\mathbf{u}(\mathbf{x}, t)$ yields also an exact, closed equation for each order N of single-time moments $\Psi(12 \cdots N) = \langle T(\mathbf{x}_1) T(\mathbf{x}_2) \cdots T(\mathbf{x}_N) \rangle$ [10–14]. The relation of the anomalous scaling we propose to the N-point equations must wait for a later paper.

Several recent preprints have addressed the question of anomalous versus regular scaling [12–14]. Chertkov *et al.* [12] study the convergence radius of an iterative expansion of the four-point moment equation and conclude that $S_4(r)$ scales regularly in the inertial range for $d \geq 3$. This conflicts with our inference from realizability considerations. Fairhall *et al.* [13] exploit summations of renormalized perturbation expansions to examine the possibility of nonperturbative effects and anomalous scaling. They find support for (11) and (12).

Equation (11) has been tested against simulations [diffuse neutron scattering (DNS)] for $d = 2$. A rapidly changing velocity field was simulated economically by the device of sweeping a frozen velocity field rapidly through the scalar field (in effect, adding a large uniform, constant velocity to the fluctuating velocity). This generates anisotropy, which was reduced to an acceptable level by sweeping two frozen velocity fields past the scalar field at right angles to each other. The full results will be presented elsewhere [7].

The simulation was done on an 8192^2 grid in a cyclic box of side 2π, using a fourth-order, purely x-space integration scheme. The initial field T was Gaussian with a spectrum $\propto k^{-3/2}$ over the range $1 < k < 1000$ so as to give a nominal initial scaling range $S_2(r) \propto r^{1/2}$. The swept velocity fields had a power-law spectrum over the range $1 < k < 1000$ chosen to give a nominal scaling range $\eta(r) \propto r^{3/2}$, corresponding to $\zeta_2 = 2 - \zeta(\eta) = 1/2$. Forcing confined to $k < 5$ maintained a statistically steady state. Equations (2) and (11) were integrated with equivalent initial conditions, parameter values, and forcing.

Figure 1 shows DNS data for $H(\Delta T)$. The curves would be lines of unit slope if (10) and (11) were exact. There is good collapse. The wiggle in H near the origin appears to arise from the low-k statistical fluctuations evident at large r in Fig. 2. If the phases of the Fourier modes of T are randomized for $k \leq 20 = 2\pi/0.314$, the wiggle is destroyed (see inset of Fig. 1). Linearity of $H(\Delta T)$ in the inertial range is supported by other numerical and experimental data for both passive scalars and thermal convection [15].

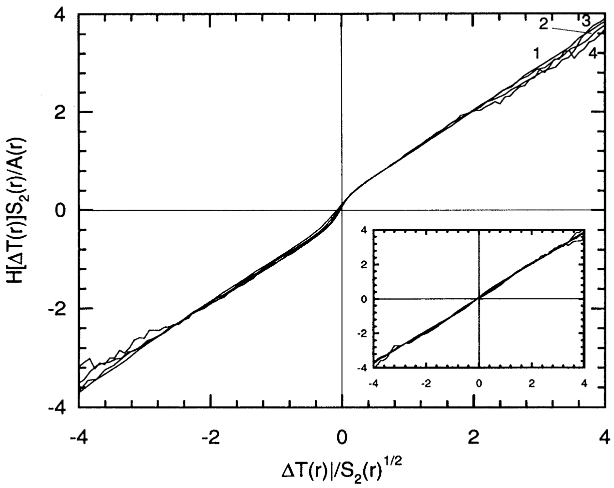

FIG. 1. Measured values of the conditional mean $H(\Delta T)$ for $r \approx 0.006$, 0.012, 0.025, 0.049 (curves 1–4, respectively); $r = 0.04$ is about the middle of the approximate inertial range in Fig. 2. The inset corresponds to randomization of phases for $k \leq 20$. Note that $A(r) < 0$.

Figure 2 compares $S_{10}(r)/945r^{\zeta_{10}}$ and $[S_2(r)]^5/r^{\zeta_{10}}$ obtained from DNS, and from (2) and (11). $S_{10}/945 = S_2^5$ for Gaussian T. At $\zeta_2 = 0.5$, (12) gives $\zeta_{10} = 1.6085$. In the range of r covered by the DNS curves, theoretical dissipation-range corrections to (11) are estimated to be negligible [7]. In our DNS, the support of $S_{10}(r) = \int_{-\infty}^{\infty} P(\Delta T, r)(\Delta T)^{10}\, d\Delta T$ for inertial-range r extends to values of $|\Delta T|/[S_2]^{1/2}$ larger than the range, shown in Fig. 1, for which we have been able to evaluate $H(\Delta T)$ directly. Good agreement between theoretical and DNS values of $\ln S_{10}(r)/[S_2(r)]^5$ suggests that the linearity effectively extends beyond the range of Fig. 1, but our understanding of this point is incomplete.

Restrictions on scaling like those for the passive scalar may also arise in Navier-Stokes (NS) turbulence. Anomalous scaling may require $\langle \nabla^2 \Delta\mathbf{u} | \Delta\mathbf{u} \rangle \propto -\Delta\mathbf{u}|\Delta\mathbf{u}|^q$ for some q. If $q = 0$, (12) may be a zeroth approximation to NS scaling. Nelkin [16] has exhibited a model formula for exponents with asymptotic square-root dependence on order. It seems to agree with high-order experimental data as well as any other current model.

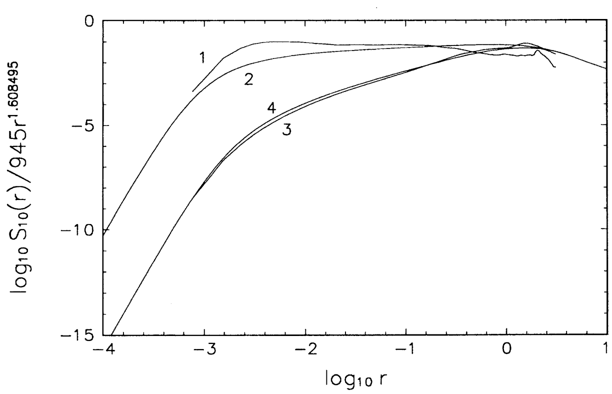

FIG. 2. $S_{10}(r)/945r^{1.6085}$ (DNS, curve 1; theory, curve 2) and $[S_2(r)]^5/r^{1.6085}$ (DNS, curve 3; theory, curve 4).

Dissipation in a Burgers-equation flow at large Reynolds number concentrates at the points of sharp sawteeth. Such structures yield an analog of (11), but a linear $\langle \nabla^2 \Delta u | \Delta u \rangle$ cannot be inferred because the ζ_{2n} are n independent [1,6]. In fact, the sawtooth model suggests $\langle \nabla^2 \Delta u | \Delta u \rangle \propto -\Delta u |\Delta u|$ for negative, inertial-range Δu. Current simulations by Gotoh [17] support nonlinearity of $\langle \nabla^2 \Delta u | \Delta u \rangle$.

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[1] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[2] H. Chen, S. Chen, and R. H. Kraichnan, Phys. Rev. Lett. **63**, 2657 (1989).
[3] Formalisms using conditional averages of $|\nabla T|^2$ were earlier introduced by S. B. Pope, Combust. Flame **27**, 299 (1976); Y. G. Sinai and Y. Yakhot, Phys. Rev. Lett. **63**, 1962 (1989).
[4] E. S. C. Ching, Phys. Rev. Lett. **70**, 283 (1993).
[5] S. B. Pope and E. S. C. Ching, Phys. Fluids A **5**, 1529 (1993).
[6] Realizability inequalities rule out increase of $\zeta_{2n+2} - \zeta_{2n}$ with n. The degenerate case $\zeta_{2n+2} - \zeta_{2n} = 0$ occurs under Burgers equation.
[7] S. Chen *et al.* (to be published).
[8] R. H. Kraichnan, J. Fluid Mech. **62**, 305 (1974).
[9] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[10] R. H. Kraichnan (unpublished).
[11] B. I. Shraiman and Eric D. Siggia, Phys. Rev. E **49**, 2912 (1994).
[12] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev (to be published).
[13] A. L. Fairhall, O. Gat, V. L'vov, and I. Procaccia (to be published).
[14] K. Gawędzki and A. Kupianen (to be published).
[15] D. Lohse (private communication); B. Castaign *et al.* (to be published).
[16] M. Nelkin (to be published). The model specializes a class due to E. A. Novikov, Phys. Rev. E **50**, 3303 (1994).
[17] T. Gotoh (private communication).