# Spatial Correlation in Quantum Chaotic Systems with Time-Reversal Symmetry: Theory and Experiment

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The correlation between the values of wave functions at two different spatial points is examined for chaotic systems with time-reversal symmetry. Employing a supermatrix method, we find that there exist long-range Friedel oscillations of the wave function density for a given eigenstate, although the background wave function density fluctuates strongly. We show that for large fluctuations, once the value of the wave function at one point is known, its spatial dependence becomes highly predictable for increasingly large space around this point. These results are compared with the experimental wave functions obtained from billiard-shaped microwave cavities, and very good agreement is demonstrated.

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Quantum properties of classically chaotic systems such as "billiards" and quantum dots are revealed to have remarkable universal behaviors that depend only on the generic symmetry of the system, such as the time-reversal symmetry and/or the spin rotational symmetry [1–3]. It has been shown that the spectral statistics are well described by universal statistical correlations derived from the random matrix theory [4,5]. (See also [6–8] for reviews.) Complementary and comprehensive information beyond the energy statistics can be obtained by examining the statistics of chaotic wave functions. For example, the distribution of the local density in a fully chaotic system is known to be universal and to obey the Porter-Thomas distribution [1,9], which is given for a system with time-reversal symmetry by the equation

$$P_0(v) \equiv \langle \delta(v - V|\psi_\epsilon(\mathbf{r})|^2)\rangle = \frac{1}{\sqrt{2\pi v}} \exp(-v/2), \qquad (1)$$

where $\psi_\epsilon(\mathbf{r})$ is the eigenfunction with energy $\epsilon$ in a system with volume $V$. $\langle\cdots\rangle$ means the average over the disorder and/or irregular potential. Equation (1) tells us that wave functions fluctuate strongly but in a universal way.

To get further understanding about the nature of chaotic wave functions, other statistical quantities that can characterize their spatial correlations are desirable and needed. The average behavior of the amplitude of the wave function has been conjectured by Berry to be similar to a pattern generated from random superposition of plane waves [10,11]. Based on this assumption the average amplitude correlations were shown to be a Bessel function. Recently, these expressions have been derived within the supersymmetry formalism [12]. In this Letter, taking a chaotic system with time-reversal symmetry, correlations about a particular value of the wave function have been derived analytically and compared to experiments for the first time. This is not only a more stringent test of the universality of chaotic wave functions, but also gives us a handle on knowing the behavior of a wave function of the system, once the wave function is known only at a limited number of points. Our main object is to investigate the joint probability distribution function of the density for two different spatial points ($r = |\mathbf{r}_1 - \mathbf{r}_2|$) defined by

$$P(v_1, v_2; r) = \langle \delta(v_1 - V|\psi_\epsilon(\mathbf{r}_1)|^2) \times \delta(v_2 - V|\psi_\epsilon(\mathbf{r}_2)|^2)\rangle. \qquad (2)$$

Although the relevant universality class for experiments that can directly observe the amplitude of wave functions, such as quantum corrals [13,14] and microwave cavity [15,16], is orthogonal, only the expression for $P(v_1, v_2)$ in the unitary case is known so far [17], because of technical difficulties. Here we evaluate $P(v_1, v_2; r)$ for the orthogonal case by finding special techniques [Eq. (20) below]. In the microwave cavity, the electromagnetic field obeys the same equation of motion as a quantum particle in a two-dimensional billiard. This enables us to make a direct comparison between the analytical results and the experimental data. The experimental data for the wave function density were obtained from thin cylindrical microwave cavities of the Sinai stadium, by using a cavity perturbation technique [16,18]. The wave function density data were earlier seen to be consistent with Eq. (1) [18], and also in agreement with the expression for density autocorrelations obtained in Ref. [12]. Once we know $P(v_1, v_2; r)$, we can also find the conditional probability

$$P_{v_1}(v_2; r) = P(v_1, v_2; r)/P_0(v_1). \qquad (3)$$

$P_{v_1}(v_2; r)$ describes the distribution of the wave function of $v_2 = V|\psi_\epsilon(\mathbf{r}_2)|^2$ at the point $\mathbf{r}_2$, provided that $v_1 = V|\psi_\epsilon(\mathbf{r}_1)|^2$ at $\mathbf{r}_1$. Here we compare the coordinate dependence of the first and the second moments of Eq. (3) between theory and experiment.

The system under consideration can be expressed by the Hamiltonian

$$H = \frac{1}{2m}\mathbf{p}^2 + U_0(\mathbf{r}) + U_1(\mathbf{r}), \qquad (4)$$

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where $U_0(\mathbf{r})$ denotes the regular part of a confining potential, and $U_1(\mathbf{r})$ is a random potential that is responsible for the chaotic dynamics—impurities or "imperfection" of the shape of the system. We take the ensemble average over $U_1(\mathbf{r})$ by the use of the supermatrix method, which reproduces the spectral correlations of Wigner-Dyson statistics [19,20] and recently was successfully applied to calculate other universal properties relating to chaotic wave functions [12,17,21,22].

We should remark that, although the supermatrix method was originally derived from the Gaussian random potential where the mean free path $\ell$ is much smaller than the size of the system $L$, the ergodicity hypothesis [23] allows us to extend our present result for cases $\ell \sim L$ by identifying the averaging over space and different states for a given sample with that over disorder. As a confirmation of the ergodicity hypothesis we will demonstrate that the theoretical dependencies for wave functions derived from a disordered system with $\ell \ll L$ are universal and describe very well experimental results for quantum billiard systems for which $\ell \sim L$.

After lengthy calculations, which we will sketch later, we have obtained the following analytical expression for $P(v_1, v_2; r)$ for the orthogonal case:

$$P(v_1, v_2; r) = \frac{1}{2\pi f(r)} \int_0^{f(r)} \frac{p\,dp}{\sqrt{f^2(r) - p^2}} \left(1 + p\frac{d}{dp}\right) \sqrt{\frac{1 - p^2}{2\pi v_1 v_2}} \int_0^{\infty} \frac{dz}{\sqrt{z}}\, e^{z/2} \varphi\left(\frac{v_1 + z}{2}, \frac{v_2 + z}{2}; p\right), \quad (5)$$

$$\varphi(v_1, v_2; p) = \frac{1}{1 - p^2} I_0\left(\frac{2p\sqrt{v_1 v_2}}{1 - p^2}\right) \exp\left(-\frac{v_1 + v_2}{1 - p^2}\right), \quad (6)$$

where $I_0(p)$ is the modified Bessel function and $f(r)$ is the Friedel function [24]. Note that $P(v_1, v_2; r)$ depends on $f(r)$ in a universal way and all coordinate dependence of $P(v_1, v_2; r)$ is incorporated only through $f(r)$. In fact, the function $f(r)$ represents the average correlation of the amplitude of the wave function [10,11]: $f(r) = V^2\langle \psi_\epsilon^*(r_1)\psi_\epsilon(r_2)\rangle$. For the case of a flat background potential in a $d$-dimensional system, $f(r)$ becomes

$$f(r) = \Gamma(d/2)\,(2/kr)^{d/2-1} J_{d/2-1}(kr) e^{-r/2\ell}, \quad (7)$$

where $k$ is the wave vector ($\epsilon = \hbar^2 k^2/2m$), $J_n(x)$ is the Bessel function, and $\Gamma(n)$ is the gamma function. Note that the envelope of $f(r)$ decays like $(kr)^{-(d-1)/2}$ for $k^{-1} \lesssim r \lesssim \ell$, and this behavior corresponds to the representation of a chaotic wave function as a random superposition of plane waves [10,11].

It should be remarked that the same distribution function in the unitary case is given by $P(v_1, v_2; r) = \varphi(v_1, v_2; f(r))$ [17]. Because of two additional integrations in Eq. (5) the spatial correlations for the orthogonal case are weaker and the fluctuations are stronger in comparison with the unitary one.

We first check that $P(v_1, v_2; r)$ given by Eq. (5) yields correct limiting behaviors. For remotely separate points such that $f(r) \approx 0$ the fluctuations of the wave function density within the given eigenstate become independent, i.e., $P(v_1, v_2; r) \approx P_0(v_1)P_0(v_2)$. In the opposite limit of close enough points that $f(r) \approx 1$, there is an obvious strong correlation between fluctuations as

$$P(v_1, v_2; r) \approx \frac{P_0(v_1)}{\sqrt{8\pi v_1(1 - f^2)}} \exp\left[-\frac{(v_1 - v_2)^2}{8v_1(1 - f^2)}\right]. \quad (8)$$

From Eq. (8), we can extract information about the gradient of the wave function. By setting $\psi(\mathbf{r}_2) = \psi(\mathbf{r}_1) + r\nabla_{\mathbf{n}}\psi(\mathbf{r}_1) + O(r^2)$, and expanding for small $r$, we obtain the joint distribution involving the wave function and its gradient in any direction $\mathbf{n} = \mathbf{r}/r$. Accordingly we find that the gradient of the wave function along any direction fluctuates independently of the value of the wave function, and obeys also the Porter-Thomas distribution,

$$\left\langle \delta(v - V|\psi(\mathbf{r})|^2)\delta\left(s - \frac{Vd}{2k^2}|\nabla_{\mathbf{n}}\psi(\mathbf{r})|^2\right)\right\rangle = P_0(v)P_0(s). \quad (9)$$

This conclusion is, however, not true for higher gradients of the wave function.

The conditional probability $P_{v_1}(v_2; r)$ is obtained straightforwardly with Eqs. (3) and (5). To see how the fluctuations of the wave functions behave and to compare between the analytical results and the experiments, the conditional average $\langle v_2\rangle_{v_1}$ and the conditional variance are more convenient. Denoting $\delta v_2 = v_2 - \langle v_2\rangle_{v1}$, we obtain

$$\langle v_2\rangle_{v_1} = 1 + f^2(r)(v_1 - 1), \quad (10)$$

$$\langle(\delta v_2)^2\rangle_{v_1} = 2 + 4f^2(r)(v_1 - 1) + 2f^4(r)(1 - 2v_1). \quad (11)$$

Comparing with results obtained for the unitary case [17], we find that the conditional average Eq. (10) is exactly the same. Thus we cannot tell the symmetry of the system only from the averaged amplitude even if we know the conditional one. To detect the symmetry, we have to examine the variance, where there is a factor of 2 difference between the orthogonal and the unitary cases.

In Figs. 1 and 2, we compare the analytical results of the conditional average and variance with the experimental data from microwave cavities. The experimental curves were obtained by picking points in a wave function with the same value and calculating the average wave function value a distance $r$ from it on a circle. This quantity was then again averaged over at least 50 wave functions after

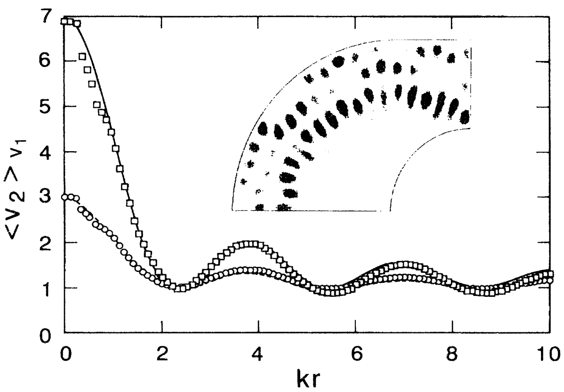


FIG. 1. The average spatial dependence of chaotic wave function squared $\langle v_2 \rangle_{v_1} = V\langle |\psi_\epsilon(\mathbf{r}_2)|^2 \rangle$ whose value $V|\psi_\epsilon(\mathbf{r})|^2$ at a point $\mathbf{r}_1$ is known to be $v_1$ [$r = |\mathbf{r}_2 - \mathbf{r}_1|$, $V$ is the volume, and $\hbar^2 k^2/(2m) = \epsilon$]. The theoretical prediction Eq. (10) is compared with experiments from the microwave Sinai stadium cavity for $v_1 = 7$ (□) and $v_1 = 3$ (○). Inset: Representative eigenfunction of the chaotic Sinai stadium.

rescaling the wave number to obtain better statistics. Figure 1 shows the plot of $\langle v_2 \rangle_{v_1}$ in Eq. (10) with experimental results for $v_1 = 2$ and $v_1 = 7$. Very good agreement is seen for both sets. In fact, agreement is excellent for all values of $v_1$ above 1, below which the noise and errors in the measurement of the wave function measurements lead to qualitative differences. In Fig. 2, the comparison of the data to the expression in Eq. (11) was done. Again one sees an excellent agreement with experimental errors of 5%, which is the level of experimental accuracy.

According to Eqs. (10) and (11), we can say, as in the unitary case [17], that large fluctuations of the wave function have some striking structure that is not present for small fluctuations. For $v_1 \gg 1$ the ratio of the variance to the average square is

$$\frac{\langle (\delta v_2)^2 \rangle_{v_1}}{\langle v_2 \rangle_{v_1}^2} \approx 2(1 - f^2) \frac{1 + 2v_1 f^2}{(1 + v_1 f^2)^2}. \qquad (12)$$

Therefore at $r \lesssim \xi$, where the "correlation length" $\xi \sim k^{-1} v_1^{1/(d-1)} \gg k^{-1}$, the variance $\langle (\delta v_2)^2 \rangle_{v_1}$ can be very small in comparison with $\langle v_2 \rangle_{v_1}^2$. It means that once we know that the wave function $|\psi(\mathbf{r})|^2$ is equal to $v_1$ at $\mathbf{r}_1$, it is highly likely to have a value $\langle v_2 \rangle_{v_1} \sim f^2 v_1$ at $\mathbf{r}_2$ for $r \lesssim \xi$. In this sense, the large fluctuation behavior of the wave function becomes highly predictable. In contrast, for small fluctuations $v_2 \ll 1$, we easily see that $\langle (\delta v_2)^2 \rangle_{v_1} \approx 2\langle v_2 \rangle_{v_1}^2$, independent of $v_1$. We also find directly from Eq. (5) that fluctuations turn out to be independent, i.e., $P_{v_1}(v_2; r) \approx 1/\sqrt{v_2}$. Although more careful evaluation gives us a correlation length $\xi$ that ensures independent fluctuations for the region $r \gtrsim \xi$, $\xi$ in such an evaluation turns out to be very small, i.e., $\xi \sim k^{-1}\sqrt{v_1} \ll k^{-1}$.

These behaviors are qualitatively the same in systems both with and without time-reversal symmetry. Therefore we can say that this is a generic property of chaotic wave functions. Also, in the semiclassical description of chaotic systems, periodic and closed orbits are known to be associated with large values of the wave functions $|\psi(\mathbf{r})|^2$ [25,26]. In this respect, our present results may imply that there is some structure present in these orbits.

Now let us proceed with the derivation of our main result in Eqs. (5) and (6). To evaluate the joint distribution $P(v_1, v_2; r)$, we work with its moments,

$$q_{nm}(r) = V^{n+m} \langle |\psi_\epsilon(\mathbf{r}_1)|^{2n} |\psi_\epsilon(\mathbf{r}_2)|^{2m} \rangle \equiv \langle v_1^n v_2^m \rangle. \quad (13)$$

$q_{nm}$ is known to be closely related to the moments of the *exact* retarded and advanced Green functions $G_\gamma^{R,A}$,

$$F_{nm}(r;\gamma) = \frac{i^{n-m}}{(\pi\nu)^{n+m}} \langle [G_\gamma^R(\mathbf{r}_1, \mathbf{r}_1)]^n [G_\gamma^A(\mathbf{r}_2, \mathbf{r}_2)]^m \rangle, \quad (14)$$

where $\nu$ is the average density of states (DOS) and

$$G_\gamma^{R,A}(\mathbf{r}_1, \mathbf{r}_2) = \sum_\alpha \frac{\psi_\alpha(\mathbf{r}_1)\psi_\alpha^*(\mathbf{r}_2)}{\epsilon - \epsilon_\alpha \pm i\gamma/2}. \quad (15)$$

We can obtain $q_{nm}$ in terms of $F_{nm}$ by the relation [27,28]

$$q_{nm} = \frac{(n-1)!\,(m-1)!}{2(n+m-2)!} \lim_{\gamma\to 0} \left(\frac{\gamma}{\Delta}\right)^{n+m-1} F_{nm}, \quad (16)$$

since the leading contribution to $F_{nm}$ for small $\gamma$ comes from the state whose energy $\epsilon_\alpha$ coincides with $\epsilon$.

$F_{nm}$ can be evaluated by the supermatrix method. However, since we cannot utilize a simple expression like Eq. (12) of Ref. [17] for the orthogonal case, we are forced to expand $F_{nm}$ directly by the Friedel function $f(r)$ as (see also Refs. [12,21])

$$F_{nm}(r;\gamma) = n!\,m! \sum_q C_q(n,m) f^{2q}(r). \quad (17)$$

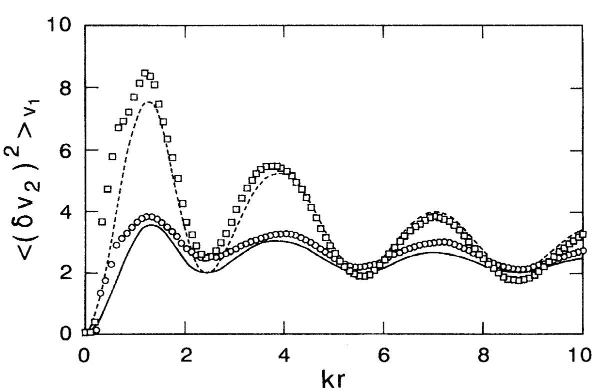


FIG. 2. Comparison of the conditional variance of a wave function $\langle (\delta v_2)^2 \rangle_{v_1}$ [$\delta v_2 = v_2 - \langle v_2 \rangle_{v_1}$ and $v_2 = V|\psi_\epsilon(\mathbf{r}_2)|^2$] as a function of the distance from point $\mathbf{r}_1$ and a reference value $v_1$ [$v_1 = V|\psi_\epsilon(\mathbf{r}_1)|^2$] between the theory [Eq. (11)] and the experiment for $v_1 = 7$ (□) and $v_1 = 3$ (○).

Using the same notation for the supermatrix elements as in Ref. [19], and defining $\tilde{Q}^{ab} \equiv (-1)^a Q^{ab}$ (for $a = 1,2$), the coefficient $C_q(n,m)$ in Eq. (17) is given in terms of $Q$-supermatrix elements:

$$C_q(n,m) = \sum \prod_{a,b=1}^{2} \frac{\langle (\tilde{Q}_{34}^{ab})^{k_{ab}} (\tilde{Q}_{43}^{ab})^{p_{ab}} (\tilde{Q}_{33}^{ab})^{l_{ab}} \rangle_Q}{(1+\delta_{ab})^{k_{ab}+p_{ab}} k_{ab}!\, p_{ab}!\, l_{ab}!}, \tag{18}$$

where the summation is taken over all the possible combinations of non-negative integers $k_{ab}$, $p_{ab}$, $l_{ab}$, which satisfy the condition $2q = \sum_{a\neq b}(k_{ab} + p_{ab} + l_{ab})$, $m - \sum_a l_{1a} = 2k_{11} + \sum_{a\neq b} k_{ab} = 2p_{11} + \sum_{a\neq b} p_{ab}$, and $n - \sum_a l_{a2} = 2k_{22} + \sum_{a\neq b} k_{ab} = 2p_{22} + \sum_{a\neq b} p_{ab}$. The symbol $\langle \cdots \rangle_Q$ denotes an integration over the saddle point manifold, i.e.,

$$\langle \cdots \rangle_Q \equiv \int DQ\, (\cdots) \exp[-(\pi\gamma/4\Delta)\mathrm{Str}\Lambda Q], \tag{19}$$

where the definitions of $\Lambda$ and Str as well as the structure of the $Q$ matrix are found in [19].

In principle, the averaging $\langle \cdots \rangle_Q$ in Eq. (18) can be carried out by using the parametrization of Ref. [19]. However, we found it technically unfeasible to evaluate this expression in such a general form. Fortunately, to get Eq. (16) one needs only know $F_{nm}(\gamma \to 0)$. The leading contribution in this limit can be extracted by transforming parameters $\lambda_i = 1 + u_i\sqrt{\Delta/\gamma}$ for $i = 1,2$ in the parametrization given in Ref. [19]. By calculating the leading order of small $\gamma$, we find the relation

$$\tilde{Q}_{cd}^{ab}\, \tilde{Q}_{c'd'}^{a'b'} \simeq \tilde{Q}_{cd'}^{ab'}\, \tilde{Q}_{cd'}^{ab'}, \tag{20}$$

for $a,b = 1,2$ and $c,d = 3,4$. Substituting Eq. (20) into Eq. (18) and combining with Eq. (16), we finally obtain after integrating over the $Q$ matrix,

$$q_{nm}(r) = \sum_q \frac{(2n-1)!!\,(2m-1)!!\,f^{2q}(r)}{2^{n+m-2q}\,(n-q)!\,(m-q)!\,(2q)!}. \tag{21}$$

Reconstructing $P(v_1, v_2; r)$ from the moments $q_{nm}$ completes the derivation of our main result Eqs. (5) and (6).

In conclusion, we have presented analytical results for universal statistical quantities that characterize the coordinate dependence of chaotic wave functions of the system with time-reversal symmetry. Further we have demonstrated excellent agreement between the theoretical results and experimental results of microwave cavities. The spatial correlations demonstrate the long-range Friedel oscillations of wave function density and the existence of extended spatial regions of high wave function density.

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