Delocalization of Electrons in a Random Magnetic Field

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Delocalization problem for a two-dimensional noninteracting electron system under a random magnetic field is studied. The Chern number is used to characterize the extended states in such a system, and by studying finite-size scaling of the density of extended states, an insulator-metal phase transition is revealed. The delocalization phase is found at the center of the energy band separated from the localized band tails by critical energies $\pm E_c$. Both the localization exponent and the critical energy E_c are shown to be dependent on the strength of the random magnetic field.

The Anderson localization theory [1,2] predicts that all states in a two-dimensional (2D) electron system are localized in the absence of a magnetic field. The quantum Hall effect (QHE) system is a first example of 2D systems that show the existence of truly extended states [3,4]. In this latter case, the presence of a magnetic field breaks the time-reversal symmetry and destroys constructive interference of the backward scattering [2] so that it is possible for electrons to propagate forwardly.

Recently, an intensive attention has been attracted to the delocalization problem in a 2D random-magnetic-field system. This problem is closely related to the half-filled QHE system [5,6] as well as the gauge-field description [7,8] of the high-T_c superconductivity problem. However, despite a lot of numerical and theoretical efforts, the issue of delocalization still remains controversial. Analytically, Zhang and Arovas [9] have recently argued that the field-theory description, which corresponds to a nonlinear sigma model of the unitary class without a topological term due to zero average of magnetic field, should have a term representing a long-range logarithmic interaction of the topological density (due to the local magnetic field). This singular term may lead to a phase transition from a localized state to an extended one. But it is contradictory to the conclusion that all the states are localized, obtained by Aronov, Mirlin, and Wolfle [10] in a similar approach. Earlier numerical works [11–13] also have given conflicting results. Recently, with a larger sample size, Liu et al. [14] have found a scaling behavior of the localization length near the energy band tail, which can be extrapolated to give an insulator-metal transition energy E_c. Nevertheless, a metallic phase has not been directly confirmed since no scaling behavior has been found there. In the possible metallic region, an even larger sample size may be needed in order to distinguish whether the states are really extended or very weakly localized [11] with the localization length much longer than the sample size.

Thus it would be desirable to study this delocalization problem from an alternative numerical method that directly probes topological properties of a system with less finite-size effect. Thouless and co-workers [15] and oth-ers [16,17] have found that delocalization property of a wave function in the presence of a magnetic field can be well characterized by its associated quantized Hall conductance. Nodes of an eigenstate wave function with nonzero Hall conductance can move freely and cover the whole real space when one continuously changes the boundary condition [17]. Such a covering of real space by the nodes has been related to a topological invariant integer (known as the first Chern number), which is identical to the quantized number of the Hall conductance (in units of e^2/h). Thus a nonzero Hall conductance describes the extensiveness of a wave function. In contrast, a zero Hall conductance state will always be localized in 2D with the presence of weak impurities (Anderson localization). In the QHE system, Huo and Bhatt [18] have calculated the boundary-phase-averaged Hall conductance for each eigenstate of a noninteracting electron system in the presence of strong magnetic field, and extrapolated the density of extended states (with nonzero Hall conductance) to the thermodynamic limit (sample size varying from 8 to 128). They have found that all extended states collapse to a single energy E_c at the center of the Landau band, with a localization length $\xi \propto 1/|E - E_c|^\nu$ and localization exponent $\nu = 2.4$ in agreement with previous known results.

In the present random magnetic field case, the total Hall conductance on average has to be zero. But one still finds nonzero quantized Hall conductances for eigenstates at each random-flux configuration. Because of the general relation between a nonzero Hall conductance and delocalization of the corresponding wave function [15], one can use this topological quantity to characterize delocalized states. A similar point of view also lies in the heart of the field-theory approach of Zhang and Arovas [9]. In this Letter, we shall use this topological property in our numerical approach. By studying the sample-size dependence of the density of extended states, which are states with nonzero Hall conductance, an insulator-metal phase transition will be revealed. The extended states are found near the center of the energy band, and the states at the band tail are all localized with both localization exponent and transition energy $\pm E_c$ depending on the random magnetic field strength.

We consider a tight-binding lattice model of noninteracting electrons under a random magnetic field. The Hamiltonian is defined as follows:

$$H = -\sum_{\langle ij \rangle} e^{ia_{ij}} c_i^\dagger c_j + \sum_i w_i c_i^\dagger c_i. \tag{1}$$

Here $c_i^\dagger$ is a fermionic creation operator, with $\langle ij \rangle$ referring to the two nearest neighboring sites. A magnetic flux per plaquette is given as $\phi = \sum_\square a_{ij}$, where the summation runs over four links around a plaquette. We study the case in which ϕ for each plaquette is randomly distributed between $-h_0\pi$ and $h_0\pi$, and w_i is also a random potential with strength $|w_i| \leq w$. For

simplicity, we assume no correlations among different plaquettes for ϕ and different sites for w_i (white noise limit). The total flux for each random configuration is always chosen to be zero. The finite system is diagonalized under the generalized boundary condition $|\Psi(i + L_j)\rangle = e^{i\theta_j}|\Psi(i)\rangle$ ($j = 1, 2$ represent x and y directions, respectively) with lattice width $L_1 = L_2 = L$, and a total number of lattice sites (sample size) is $\mathcal{N} = L \times L$ (the lattice constant is chosen to be the unit).

The Hall conductance can be calculated by using the Kubo formula. One may relate a Hall conductance to each eigenstate $|m\rangle$,

$$\sigma_{xy}^{(m)} = \frac{ie^2 h}{2\pi} \sum_{n \neq m} \frac{\langle m | p_x | n \rangle \langle n | p_y | m \rangle - \langle m | p_y | n \rangle \langle n | p_x | m \rangle}{(\varepsilon_m - \varepsilon_n)^2}, \tag{2}$$

where $\mathbf{p}$ is the velocity operator defined as $p_\tau = i\sum_i(c_{i+\tau}^\dagger c_i e^{ia_{i+\tau,i}} - c_i^\dagger c_{i+\tau} e^{-ia_{i+\tau,i}})$ with $\tau = \hat{x}$ or $\hat{y}$. The total Hall conductance for the system is given by $\sigma_H = \sum_{\varepsilon_m < \varepsilon_F} \sigma_{xy}^{(m)}$ at zero temperature, with ε_F as the Fermi energy. σ_H will always be zero on average in the case of a random magnetic field. However, $\sigma_{xy}^{(m)}$ can be nonzero for each random-flux configuration because of the breaking of time-reversal symmetry. As pointed out before, a state with nonzero (quantized) Hall conductance represents an extended state in the thermodynamic limit,

whereas a zero Hall conductance state should always be localized in 2D.

Direct calculation of matrix elements in formula (2) is time consuming. We can make a unitary transformation $|\Phi\rangle = e^{-i\theta_1 x/L} e^{-i\theta_2 y/L}|\Psi\rangle$ such that the new state $|\Phi\rangle$ satisfies a periodical boundary condition. The Hamiltonian (1) is transformed in terms of $a_{ii+} \rightarrow a_{ii+\tau} + \theta_\tau/L$ with $\tau = 1$ ($\hat{x}$) or 2 ($\hat{y}$). Then the boundary-phase averaged conductivity can be related to partial derivatives of the wave function in the following form [15]:

$$\sigma_{xy}^{(m)} = \frac{ie^2}{4\pi h} \oint d\theta_j \sum_i \left(\Phi_m^*(\theta_1, \theta_2; i) \frac{\partial \Phi_m(\theta_1, \theta_2; i)}{\partial \theta_j} - \frac{\partial \Phi_m^*(\theta_1, \theta_2; i)}{\partial \theta_j} \Phi_m(\theta_1, \theta_2; i) \right), \tag{3}$$

where the closed path of the integral is along the boundary of a unit cell $0 \leq \theta_1, \theta_2 \leq 2\pi$. $\sigma_{xy}^{(m)}$ in (3) can be shown [15,16] to be quantized in units of e^2/h. Here Φ is required to be an analytic wave function in 2D θ space. Starting from the wave function $\Phi(0, 0; i)$ defined at one corner of the boundary in θ space, the phase of the wave function Φ can be uniquely determined [15] by a process of parallel translation, first along the θ_1 axis and then along the θ_2 axis as

$$\sum_i \Phi^*(\theta_1, 0; i) \frac{\partial \Phi(\theta_1, 0; i)}{\partial \theta_1} = 0, \tag{4a}$$

$$\sum_i \Phi^*(\theta_1, \theta_2; i) \frac{\partial \Phi(\theta_1, \theta_2; i)}{\partial \theta_2} = 0. \tag{4b}$$

Numerically we have diagonalized the Hamiltonian with the boundary angle varying in whole $2\pi \times 2\pi$ phase space for each given random flux and potential configuration. At each step, θ_j may only change by a very small value such that $\partial\Phi/\partial\theta$ can be well approximated by $\Delta\Phi/\Delta\theta$ (usually $\Delta\theta < 2\pi/100$, which is adjustable in our numerical calculation to give a reliable result). By constructing a wave function satisfying conditions (4), the Hall conductance averaged over the boundary angle

is determined in terms of (3) for each eigenstate. An eigenstate with nonzero Hall conductance is defined as an extended state, and the corresponding density of states $\rho_{ext}(\varepsilon, \mathcal{N})$ is obtained as a function of energy ε and sample size $\mathcal{N}$, which is averaged over random flux-potential configurations (200–2000 random configurations depending on sample size).

The total density of states $\rho(\varepsilon, \mathcal{N})$ and the extended one $\rho_{ext}(\varepsilon, \mathcal{N})$ are obtained as a function of energy ε and lattice size $\mathcal{N}$ ($\mathcal{N} = 16, 36, 64, 100,$ and 144). The total density of states does not change much with lattice size, but the extended part of the density of states shows distinctive behaviors at different energy regions separated by critical energies $\pm E_c$. The ratio $\rho_{ext}(\varepsilon, \mathcal{N})/\rho(\varepsilon, \mathcal{N})$ is presented in Fig. 1 around $-E_c$ (with random magnetic field and impurity strengths chosen as $h_0 = 0.6\pi$ and $w = 1.0$, respectively). All the curves in Fig. 1 cross at a fixed point $\varepsilon = -E_c$, which is independent of the lattice size within the error bars. At energy $\varepsilon < -E_c$, the extended-state density is continuously suppressed, and can be extrapolated down to zero as lattice size becomes infinity (see below). On the other hand, in the regime $-E_c < \varepsilon < E_c$, ρ_{ext}/ρ monotonically increases with lattice size and eventually saturates. Therefore, Fig. 1 clearly shows a metal-
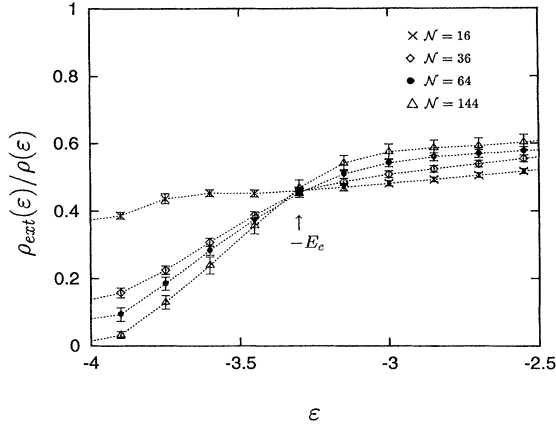
FIG. 1. The ratio of the density of extended states over the total density of states is plotted as a function of energy (around $-E_c = -3.3$) for a different sample size $\mathcal{N}$.

insulator transition at critical energies $\pm E_c$ (the curves in Fig. 1 are symmetric about $\varepsilon = 0$).

Let us consider in detail the localization at the band tail $\varepsilon < -E_c$. One may define two quantities characterizing the localization effect: a ratio R_0 of the number of extended states divided by the total number of states at energy $\varepsilon < -E_c$ region, and a mean width ΔE of the extended states in such a regime, both of which presumably will approach to zero in the thermodynamic limit. Here $R_0 = \int_{-\infty}^{-E_c} \rho_{\text{ext}}(\varepsilon)\, d\varepsilon / \int_{-\infty}^{-E_c} \rho(\varepsilon)\, d\varepsilon$ and $(\Delta E)^2 = \int_{-\infty}^{-E_c} [\varepsilon - (-E_c)]^2 \rho_{\text{ext}}(\varepsilon)\, d\varepsilon / \int_{-\infty}^{-E_c} \rho_{\text{ext}}(\varepsilon)\, d\varepsilon$. R_0 and ΔE versus the sample size are shown in Fig. 2 in a log-log plot at $h_0 = 0.6$ and $w = 1.0$. The data follow two parallel straight lines nicely, suggesting the following power-law behavior: $R_0 \sim \mathcal{N}^{-x}$ and $\Delta E \sim \mathcal{N}^{-x}$, with $x = 0.2 \pm 0.02$. Such a scaling law ensures the absence of the extended states outside the energy range $(-E_c, E_c)$ in the thermodynamic limit. In the localized region, the localization length is a characteristic length scale (scaling parameter [19]), and for a finite-size sample with a width L the states with a localization length $\xi > L$ should appear as extended ones. If the localization length goes as $1/|\varepsilon - (-E_c)|^\nu$ when $\varepsilon \to -E_c$, one expects $1/(\Delta E)^\nu \sim L$, or $\Delta E \sim \mathcal{N}^{-1/2\nu}$ (Ref. [18]). One also has $R_0 \propto \Delta E \rho(-E_c, \mathcal{N}) \propto \Delta E$. So the finite-size scalings of R_0 and ΔE in Fig. 2 imply a power-law behavior of the localization length ξ: $\xi \propto 1/|\varepsilon - (-E_c)|^\nu$ with $\nu = 1/2x = 2.5 \pm 0.3$.

At $-E_c < \varepsilon < E_c$, a monotonic increase of ρ_{ext}/ρ with sample size is manifestly metallic behavior. It is consistent with the behavior of $d(\xi_L/L)/dL > 0$ (ξ_L is the so-called decay length and ξ_L/L describes the extensiveness of the system) found in the metallic region of the 3D system and the 2D system with spin-orbit interaction (symplectic class) [19,20]. In the present approach, the quantity ρ_{ext} directly characterizes extended states and can be extrapolated to a finite value at
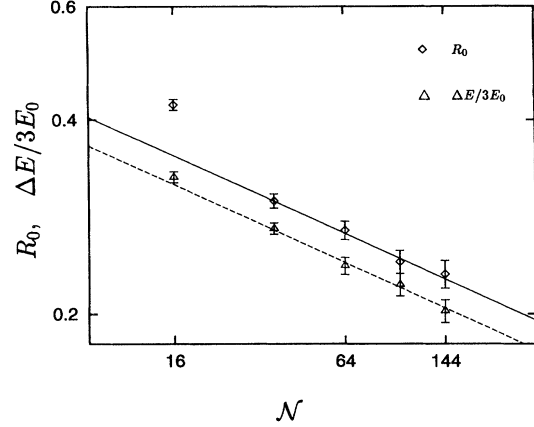

FIG. 2. R_0, the number of extended states divided by the total number of states at ($\varepsilon < -E_c$), and ΔE, the mean width of the band of extended states in the same energy region, vs sample size $\mathcal{N}$ on a log-log scale. E_0 is a mean width of the band of total states at ($\varepsilon < -E_c$). All the data are fitted into two parallel straight lines.

a large sample size limit. One may also define a ratio $N_{\text{ext}}/N_0 = \int_{-E_c}^{E_c} \rho_{\text{ext}}\, d\varepsilon / \int_{-E_c}^{E_c} \rho\, d\varepsilon$, namely, the total number of extended states divided by the total number of states within $(-E_c, E_c)$. N_{ext}/N_0 is found to saturate to a finite value $R_c \sim 0.68$ in the following manner: $N_{\text{ext}}/N_0 - R_c \propto -N^{-y}$ ($y \sim 0.3$), which is shown in Fig. 3 by a log-log plot ($h_0 = 0.6$, $w = 1.0$). A finite R_c in the thermodynamic limit is a direct evidence for delocalization. ($R_c < 1$ in the thermodynamic limit suggests that eigenstates with a nonzero Chern number will become degenerate with eigenstates of a zero Chern number. But couplings between them should make all the states delocalized at the same energy [21]. Thus, the $R_c \neq 0$ region in the thermodynamic limit should represent a
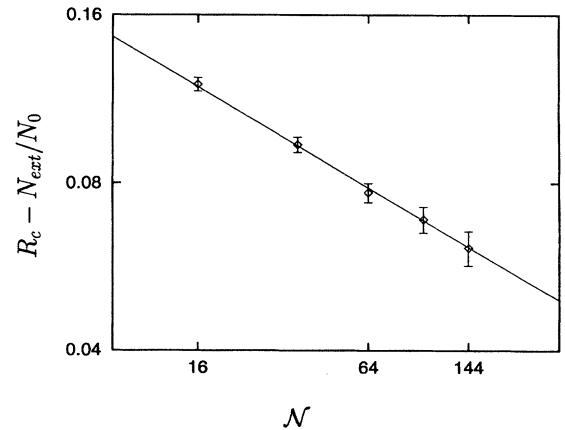

FIG. 3. N_{ext}/N_0, the number of extended states over the total number of states within the energy region ($-E_c, E_c$), vs sample size $\mathcal{N}$. The ratio is extrapolated to a finite number $R_c = 0.68$ in the thermodynamic limit, and solid line is a best fit to the data.

fully delocalized region.) The parameter R_c plays a similar role as the inverse of the effective mass renormalization factor [22], which has been argued to be a better scaling parameter than conductivity for the discussion of low-dimensional Anderson-localization problem [22]. Whether this delocalization transition is a Kosterlitz-Thouless type with the power-law decay of correlation functions, as predicted by Zhang and Arovas [9], is not clear here and needs a further investigation. Recently, the fluctuations of the Hall conductance in a random-magnetic-field system have been studied [23], and the root mean square for such fluctuations is predicted to be in the order of e^2/h. We have found $\langle \sigma_H^2 \rangle \sim 0.4(e^2/h)^2$, which is indeed consistent with the analytic analysis.

Very similar behaviors have also been obtained at other random-flux strengths: $h_0 = 0.4$ and 0.5 (with $w = 1.0$). Correspondingly, $E_c = \pm 3.7$ and ± 3.5, while $\nu = 1.25 \pm 0.3$ and 1.75 ± 0.3, respectively. The results suggest a nonuniversal localization exponent ν, which increases with h_0 and is consistent with a larger ν (~ 4.5) obtained at $h_0 > 0.7$ in Ref. [14]. The reduction of the metallic region $(-E_c, E_c)$ indicates that the extended states are less favorable at larger h_0. With the increase of h_0 we find that ρ_{ext} becomes less sensitive to the sample size. When $h_0 > 0.7$, $\partial \rho_{ext}/\partial N$ is relatively small around the center of the energy band and a larger lattice size is needed in order to get conclusive results about delocalization. We also have checked the effect of the disorder strength w on the delocalization. It turns out that the delocalization is monotonically weakened with the increase of w, as expected, and is strongly suppressed beyond $w = 2$. With the decrease of w, however, many more computation steps are needed on the average over the boundary-phase space in order to reach the exact quantization of the Chern number.

Since we study the density of states for the extended states characterized by the topological properties of wave functions (Hall conductances), the finite-size effect is expected to be less important in comparison with other approaches. The reason is that as a *boundary-condition averaged* Hall conductance, the Chern number becomes less sensitive to the detailed distribution of eigenstates. The latter, in contrast, usually has a stronger size dependence and plays an essential role in other physical quantities like the Green's functions. The existence of the fixed points $\pm E_c$, which are independent of lattice size within the error bars, as well as the finite-size scalings on two sides of $\pm E_c$ indeed support this expectation.

In conclusion, we have unambiguously demonstrated the existence of a delocalization region for a noninteracting 2D electron system under a random magnetic field. The critical energy E_c of a metal-insulator transition has been determined. Two branches of finite-size scaling are found in both metallic and localized regions, and the results are extrapolated to the thermodynamic limit. The localization length at the band tail ($\varepsilon < -E_c$, $\varepsilon > E_c$)

behaves like $\xi \sim 1/|\varepsilon \pm E_c|^\nu$, with both E_c and ν varying with the strength of the random magnetic field.

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