Hydrodynamic Description of Granular Convection

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We present a hydrodynamic model that captures the essence of granular dynamics in a vibrating bed. We carry out the linear stability analysis and uncover the instability mechanism that leads to the appearance of convective rolls via a supercritical bifurcation of a bouncing solution. We also explicitly determine the onset of convection as a function of control parameters and confirm our picture by numerical simulations of the continuum equations.

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Granular materials in a container subjected to vertical vibrations display interesting nonlinear dynamical behaviors [1–5]. Nothing really happens for $\Gamma = A\omega^2/g < 1$, where A and ω are the amplitude and the frequency of the oscillations and g is the gravitational constant. For $1 < \Gamma$, however, the granular materials collectively move up and down, which we term uniform bouncing [6], until Γ reaches the critical value Γ_c beyond which such a uniform bouncing motion becomes unstable and permanent convective rolls develop inside the bulk [3–5]. Recent studies have revealed further complexity of this problem for values of Γ much larger than 1, where, e.g., bubble formation [5] has been observed. Current efforts to understand the experiments of granular dynamics [3,5] have mostly focused on large scale molecular dynamics (MD) simulation [4]. While successful in reproducing convection cells and some of the experimental results, such studies have limitations in understanding the analytic structure of the instability mechanism and/or its subsequent dynamic evolution. There have been a handful of attempts in the past to derive continuum equations for granular dynamics notably by Jenkins and Savage for rapid granular flow problems [7] and by Haff for vibrating beds [8], but these studies have been mostly confined either to simple cases of one-dimensional oscillations in an infinite system, where pressure inside the grains has a hydrodynamic pressure gradient in steady states [8], or to cases where an explicit assumption has been made regarding the Gaussian velocity distribution of grains [7], which has been shown to break down in a dense granular system [9]. There also has been a recent attempt by Bourzutchky and Miller [10] who have utilized the Navier-Stokes equation along the similar lines of Haff [8] and have reproduced numerically convective rolls. However, we find it difficult to imagine that the hydrodynamic pressure term ($\rho g z$) exists to cancel the gravity term inside the granular materials that undergo vertical vibrations.

The purpose of this Letter is twofold: We first propose a dynamic model that is simple enough to make progress in analytic studies, yet captures, in our opinion, the essence of the granular dynamics of vibrating beds. Second, we demonstrate that the correct way of studying the convective instability is to carry out the stability analysis around the *bouncing solution* and explicitly determine the onset of convection as a function of external parameters. We will also present numerical results to confirm our predictions.

Equations of Motion.—Our starting point is the recognition that the most fundamental aspect of the vibrating bed, apart from the obvious fixed bed solution with no external driving, is the existence of a uniform bouncing of a collection of particles. Such a bouncing solution can be either a solid block inside the bed or a fluidized state with a slightly expanded volume yet with no internal degrees of freedom. This assumption is consistent with observations in MD [4], where surface fluidization rapidly spreads out into bulk regions when surface fluidization is suppressed. In such a case, the bouncing solution can be represented by a motion of an elastic ball on a vibrating platform. For small Γ, no exotic motion such as chaotic motion is expected to occur for such a ball [11]. We further assume that the restitution constant of "the center of mass" in the collection of particles (or granular block) is zero, because the random motion of particles inside the granular block may suppress the systematic elastic behavior. In such a case, the relative position of the ball with respect to the bottom plate $\Delta(t)$ is given by

$$\Delta(t) = \Gamma(\sin t_0 - \sin t) + \Gamma\cos t_0(t - t_0) - \tfrac{1}{2}(t - t_0)^2 \tag{1}$$

in the units of $g = \omega = 1$, where the ball starts to bounce at t_0 on the bottom plate, whose position at time t is given by $\Gamma\sin t$ in the experimental frame. The bouncing solution is then described by the relative speed between the plate and the ball: $V_{\mathrm{rel}} \equiv d\Delta(t)/dt$. Since $\Delta(t)$ cannot be negative, the ball launched upward on the plate at t_0 falls back to the plate at t_1 [i.e., $\Delta(t_1) = 0$] and stays there until $t = t_0 + 2\pi$ from our assumption of the zero restitution constant. The ball is then relaunched and obeys (1) again. For later use, we determine $(t_0 = \sin^{-1}(1/\Gamma), t_1)$ for different values of Γ. For example, $(t_0, t_1) = (1.181, 2.882\,25)$ for $\Gamma = 1.1$

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and $(t_0, t_1) = (0.524, 5.18)$ for $\Gamma = 2.0$. When we expand $\Delta(t)$ around t_0 we obtain $\Delta(t) \simeq (\Gamma/6)\cos(t_0)(t - t_0)^3 > 0$, where $\cos(t_0) > 0$ from the launching condition $d^2V_{\rm rel}/dt^2 > 0$. Hence there is no solution of $\Delta(t) = 0$ around $t = t_0$ except for $t = t_0$. Therefore the bouncing motion starts from a finite $t_1 - t_0$. One can now readily derive the equation of motion for the vertical coordinate z for the bouncing motion of a granular block: $\ddot{z} = (-1 + \Gamma\sin t)\theta(-1 + \Gamma\sin t)$, where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for otherwise.

In order to describe the motion of a granular block in the presence of internal degrees of freedom, such as rotation and/or translation, we define two coarse-grained dynamical variables: the density $\rho(\mathbf{r}, t)$ and the velocity $\mathbf{v}(\mathbf{r}, t)$ of the granular system. In a box fixed frame, ρ and $\mathbf{v}$ then should satisfy the continuity and the momentum equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad (2)$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \hat{z}(\Gamma\sin t - 1 - \lambda) - \frac{1}{\rho}\nabla P + \frac{1}{R}[\nabla^2 \mathbf{v} + \chi \nabla(\nabla \cdot \mathbf{v})], \qquad (3)$$

where $\hat{z}$ is the unit vector in the vertical direction and λ is a Lagrange multiplier. $\lambda = 0$ for free motion and $\lambda = \Gamma\sin t - 1$ for the stationary state. Note that the first term in (3) is due to the uniform bouncing and the third term is the energy dissipation effectively represented by the Reynolds number R and the bulk viscosity χ. Now the exact form of the pressure P in (3) is unknown for granular materials. Unlike fluid, for granular materials in a container supported by the side walls the pressure inside the bulk seems to saturate [1,12]. In such a case, the only contribution to the granular pressure would result from the hard sphere repulsion, which might be effectively represented by the van der Waals equation

$$P = \frac{T\rho}{1 - b\rho}, \qquad (4)$$

where $T \approx \langle \mathbf{v}^2 \rangle$ is the granular temperature [8] and b is a constant of order unity. Note that Eqs. (2) and (3) are precisely the *compressible* Navier-Stokes equation with two modifications: first, the hydrodynamic pressure term is absent and is replaced by the van der Waals form (4), and second, the gravity term thus survives in the vibrating bed and has been effectively modified to $g - A\omega^2 \sin t$ in physical units. Notice that the term $\Gamma \sin t$ appears since we have used the box-fixed frame. We now analyze Eqs. (2) and (3).

Linear stability analysis.—(a) Fixed bed solution: a fixed bed is a container filled with grains with no external driving. In this case, the contact force balances out gravity and the net force acting on each grain is zero. So, we use $\lambda = \Gamma\sin t - 1$ in (3). In this case, the solution with constant density $\rho = \rho_0$ and zero speed $\mathbf{v} = 0$ is stable.

(b) Linear stability of a uniform bouncing solution: In order to discuss the stability of the uniform bouncing solution, $\rho = \rho_0$ and $\mathbf{v} = (0, 0, V_{\rm rel}(t))$, against fluctuations, we set $\rho = \rho_0 + \rho_L$ and decompose the velocity into the vertical and horizontal components, $\mathbf{v}_L = (\mathbf{v}_{\perp,L}, w_L)$ with $\mathbf{v}_\perp = \mathbf{v}_{\perp,L}$ and $w = V_{\rm rel}(t) + w_L$. We then substitute these into dynamic equations (2) and (3) and introduce a new coordinate to simplify the problem, $\xi = z - \int^t V_{\rm rel}(t')\,dt'$. Upon linearization, we obtain the following equation for the perturbed density ρ_L:

$$\left[\partial_t^2 - \frac{1}{\hat{R}}\nabla_\xi^2 \partial_t - T_e \nabla_\xi^2\right]\rho_L = 0, \qquad (5)$$

where $\hat{R} = R/(1 + \chi)$. We now solve (5) in two dimensions under the no current boundary condition at the plate and at $z = \infty$, namely, $\rho_L = 0$ at $x = 0, L$, and $z = 0, \infty$, where L is the dimensionless size of the box. To satisfy these boundary conditions, we set

$$\rho_L(x, y, z, t) = \rho_{L,q,m}(t)\sin[\hat{\pi} m x]\sin[q(\xi - S(t))], \qquad (6)$$

where $\hat{\pi} = \pi/L$, m is an integer, and $S(t) = -\Delta(t)$. We notice that the spectrum is discrete for the x direction but continuous along the z direction. We now substitute (6) into (5) and utilize the fact $t = \tau + t_0$ with $t_0 = \sin^{-1} 1/\Gamma$. After some algebra, we obtain the following second order ordinary differential equation for the amplitude $\rho_q(t) = \rho_{L,q,m}(t)$:

$$\ddot{\rho}_q + B(q)\dot{\rho}_q + iC(q)\dot{\rho}_q + D(q)\rho_q + iE(q)\rho_q = iL_q(\tau)\dot{\rho}_q + M_q(\tau)\rho_q + iN_q(\tau)\rho_q, \qquad (7)$$

where

$$B(q) = \hat{R}^{-1}(\hat{\pi}^2 m^2 + q^2), \qquad C(q) = 2q\sqrt{\Gamma^2 - 1}, \qquad (8)$$

$$D(q) = T_e(q^2 + \hat{\pi}^2 m^2) - \tfrac{3}{2}q^2\Gamma^2 + q^2,$$
$$E(q) = -q + \sqrt{\Gamma^2 - 1}\hat{R}^{-1}q(\hat{\pi}^2 m^2 + q^2), \qquad (9)$$

and the time dependent inhomogeneous terms are $L_q(\tau) = 2q[\tau + \sqrt{\Gamma^2 - 1}\cos\tau - \sin\tau]$, $M_q(\tau) = -2q^2(\Gamma^2 - 1) \times \cos\tau + 2q^2\sqrt{\Gamma^2 - 1}\sin\tau + [q^2(\Gamma^2 - 2)/2]\cos(2\tau) - q^2\sqrt{\Gamma^2 - 1}\sin(2\tau) - 2q^2\sqrt{\Gamma^2 - 1}\tau(1 - \cos\tau) - 2q^2\tau \times \sin\tau + q^2\tau^2$, and $N_q(\tau) = -q\sqrt{\Gamma^2 - 1}\sin\tau - q\cos\tau + \hat{R}^{-1}q(q^2 + \hat{\pi}^2 m^2)(\tau + \sqrt{\Gamma^2 - 1}\cos\tau - \sin\tau)$.

Note Eq. (7) is valid only between $\tau = 2n\pi$ and $\tau = \tau_0 + 2n\pi$ with $\tau_0 \equiv t_1 - t_0$, during which time grains are launched from the plate by vibrations and then undergo free fall. Except for this region, it is easy to show $S(t) = 0$ and $C(q) = E(q) = L_q(\tau) = M_q(\tau) = N_q(\tau) = 0$ with $D(q) \to D_0(q) = T_e(q^2 + \hat{\pi}^2 m^2)$.

The rest of the paper is devoted to discuss the condition for the linear stability of (7) and to test numerically the validity of our approximations. We may be able to obtain an explicit solution of Eq. (7) with the aid of the assumption that the most unstable mode is the only relevant mode. The condition for instability in this treatment, however, is complicated and time dependent. In addition, this condition is not adequate for our purpose, since we are interested in the behavior for times longer than one vibrating oscillation. Therefore we may replace $L_q(\tau)$ [and $M_q(\tau)$ and $N_q(\tau)$ as well] by its average value over a *flying time* $\langle L_q(\tau)\rangle$, namely, $L_q(\tau) \simeq \langle L_q\rangle = \tau_0^{-1}\int_0^{\tau_0} d\tau\, L_q(\tau)$, and so on for $M_q(\tau)$ and $N_q(\tau)$. Equation (7) then reduces to a second order ordinary differential equation with constant coefficients. Assuming $\rho_q \sim e^{\sigma t}$, it becomes easy to obtain the eigenvalues σ for the flying motion as

$$\sigma_\pm = -\frac{B + i\tilde{C}}{2} \pm \frac{1}{2}\sqrt{(B + i\tilde{C})^2 - 4(\tilde{D} + i\tilde{E})}, \tag{10}$$

where $\tilde{C} = C(q) - \langle L_q\rangle$, $\tilde{D} = D(q) - \langle M_q\rangle$, and $\tilde{E} = E(q) - \langle N_q\rangle$. The relevant branch is σ_+ and the eigenvalues reduce to $\sigma_\pm = -B/2 \pm \frac{1}{2}\sqrt{B^2 - 4D_0}$ for stationary states.

The averaged instability condition over one oscillation cycle is then the average of $\mathrm{Re}[\sigma] > 0$. For this purpose, we introduce a function

$$\sigma_{\mathrm{eff}}(q) = \tau_0\left\{\left(\tilde{E} - \frac{B\tilde{C}}{2}\right)^2 - B^2\left(\tilde{D} + \frac{\tilde{C}^2}{4}\right)\right\} + (2\pi - \tau_0)(-B^2 D_0), \tag{11}$$

where the first term is the instability condition for (10) multiplied by the time period τ_0 in which particles can move freely [13], while the second term is that with $S(t) = 0$. If the function $\sigma_{\mathrm{eff}}(q) > 0$ for any q, it signals the instability of the uniform bouncing solution. For the finite system, $\sigma_{\mathrm{eff}}(0) = -2\pi^7 T_e/\hat{R}^2 L^6 < 0$. Thus convection disappears for infinite systems, which agrees with MD simulations [1,3,14]. Equivalently, convection also disappears in the limit of large R, i.e., either the particles are too smooth or the kinetic energy is too small to provide the necessary driving force among grains. The set of parameters that corresponds to physical situations might be $\hat{R} \sim 2$, $T_e \sim 3$, and $L = 10$, because (i) the linear size of the box $L_r = Lg/\omega^2 \simeq 0.6$ cm for $\omega/2\pi \simeq 20$ Hz, (ii) $T \sim \tau_0^{-1}\int_0^{\tau_0} V_{\mathrm{rel}}^2(\tau)d\tau \sim 3$, (iii) the kinetic viscosity for granular fluid is evaluated by $\nu_s \simeq 5 \times 10^{-3}$ m^2/s [2] and the definition of $\hat{R} = UL_r/\nu_s \sim 2$ with the aid of the characteristic velocity $U \sim \sqrt{V_{\mathrm{rel}}^2\, g/\omega} \sim 10$ cm/sec in physical units. For pure numerical reasons, however, we chose $\hat{R} = T_e = 10$ and $L = 10$. For these parameters, we first solved $\Delta(t) = 0$ numerically to determine t_1, and then computed $\sigma_{\mathrm{eff}}(q)$ as a function of q. As demonstrated in Fig. 1, $\sigma_{\mathrm{eff}}(q)$ is convex and thus has a maximum σ_m at a particular value of q. For $\Gamma \simeq 1$, $\sigma_m < 0$ and thus $\sigma_{\mathrm{eff}}(q) < 0$ for all q and the bouncing solution is stable. As we increase Γ further to the critical value, Γ_c, σ_m moves upward, crosses zero and becomes positive, in which case $\sigma_{\mathrm{eff}}(q) > 0$ for a band of q. In this case, the bouncing solution becomes unstable and we expect convective rolls to appear. The onset of convection is then determined by setting, $\sigma_m(\Gamma_c) = 0$. For $L = 10, R = T_e = 10$, we find $\Gamma_c \simeq 1.12$, and the selected wave number is about $q_c = 0.22$. The most unstable wave number q_m gradually shifts with the increase of Γ. We now check the validity of our picture by numerical simulations.

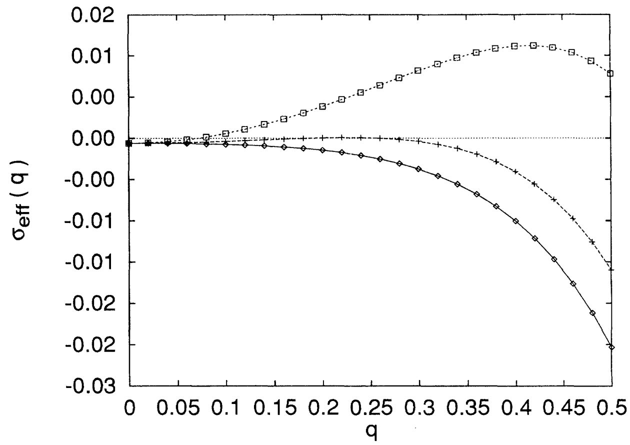

FIG. 1. The effective growth rate $\sigma_{\mathrm{eff}}(q)$ as a function of the wave number q for $\Gamma = 1.05$ (diamonds), for which $\sigma_{\mathrm{eff}}(q) < 0$ for all values of q. For $\Gamma = 1.2 > \Gamma_e = 1.12, \sigma_{\mathrm{eff}}(q)$ becomes positive for a band of q (squares). Γ_c is determined by the condition that the maximum of $\sigma_{\mathrm{eff}}(q)$ becomes zero at Γ_c (crosses). The parameters used are $T_e = R = 10$ and $L = 10$.

Numerical results.—We have solved (2)–(4) numerically in two dimensions with no slip boundary conditions at the side walls as well as at the top and bottom plates. Note that the top plate suppresses the complicated surface motion of vibrating beds and allows us to use our simplified picture. Since the granular fluid is confined in a box, we do not introduce λ explicitly in the simulations. As a result, $S(t) \approx 0$ after a grain lands on a plate in the average bouncing state. The absence of λ and the presence of the top wall is expected to cause the bouncing solution to appear even for $\Gamma \leq 1$ in contrast to the real situation. Since, however, the linearized equation (7) with $S(t) = 0$ is identical to that with nonzero λ, omitting λ would not change the essence of the dynamics. In the same spirit, we have ignored χ and b in Eqs. (3) and (4) in our simulations. Our simulation results are presented in Fig. 2 for two different values of Γ, $\Gamma < \Gamma_c$ and $\Gamma > \Gamma_c$. In the former case, the bouncing solution is expected to appear inside the bed and the density and the velocity at a given point oscillates with the same frequency as the vibration [Fig. 2(a)]. Upon increasing Γ further to $\Gamma = 1.2$, which is beyond the predicted $\Gamma_c = 1.12$ determined by (11), we find that the bouncing solution has disappeared and per-

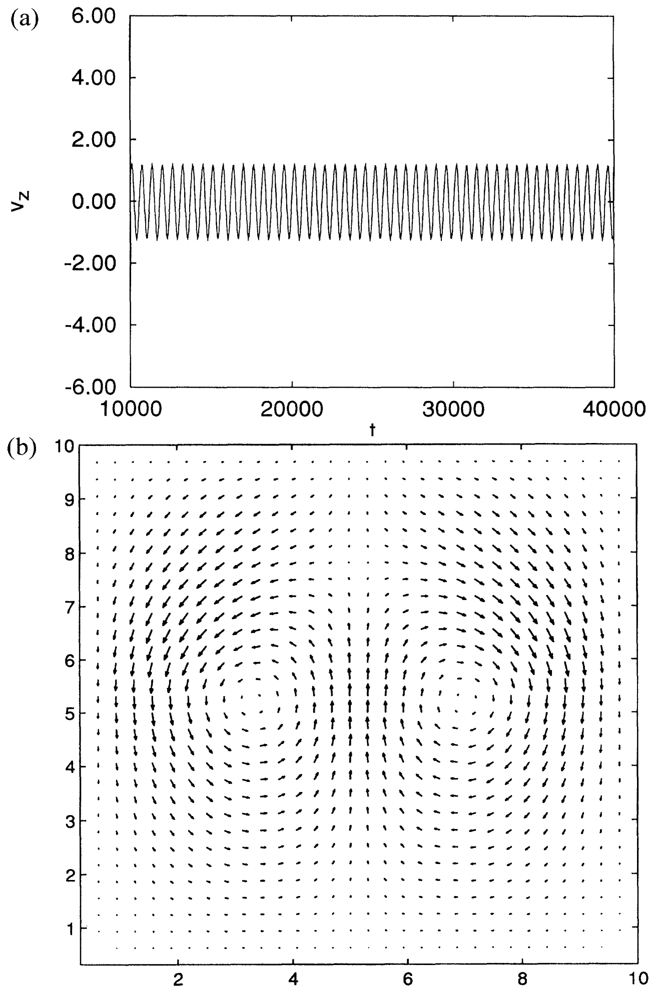

FIG. 2. (a) A bouncing solution. The speed v_z at a given point is plotted as a function of time for $\Gamma = 0.9$. (b) For $\Gamma = 1.2 > \Gamma_c = 1.12$, the bouncing solution becomes unstable and permanent convective rolls appear inside the box. The arrows are the velocity vectors and the direction of the arrow is the direction of the flow. The parameters used in simulations for (a) and (b) are the same as those in Fig. 1.

manent convective rolls have developed inside the bulk [Fig. 2(b)]. The wavelength of the most unstable mode given by the linear stability analysis is about $q_m \approx 0.4$, which is not far from the actual wavelength of the convective rolls: $q = 2\pi/\lambda = 2\pi/L \approx 0.6$.

In passing, we briefly mention the difference between the granular beds and water beds. The latter is shown to exhibit the Faraday instability at the air-water interface [15]. The crucial difference between these two systems lies in the pressure term: For the water bed, since water is incompressible, the hydrodynamic pressure term ρgz precisely cancels the gravity term in the fluid equation, thus supressing motion inside the bulk, while the absence of the hydrodynamic pressure term produces the convective instability in the bulk for the granular beds. We will present the details of our analysis including the weakly nonlinear analysis elsewhere, which will highlight the differences between the two.

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