Novel Arbitrary-Amplitude Soliton Solutions of the Cubic-Quintic Complex Ginzburg-Landau Equation

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We propose a method for finding stationary pulse solutions of the complex Ginzburg-Landau equation, which can be applied to both cubic and quintic models. In particular, we discover previously unknown arbitrary-amplitude pulse solutions for both the cubic and quintic models. Numerical simulations show that these solutions are stable relative to small perturbations.

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The complex Ginzburg-Landau equation (CGLE) is known in many branches of physics, including fluid dynamics [1], nonlinear optics [2,3] and laser physics [4,5], theory of phase transitions [6], etc. This equation is rather general, as it includes dispersive and nonlinear effects, in both conservative and dissipative forms. The CGLE possesses a rich variety of solutions, including coherent structures such as pulses, fronts, sinks, and sources [7–15], periodic and quasiperiodic solutions [16], and a transition to chaos [17]. In this paper, we concentrate on the pulse solutions, as they are the most important for many applications. We call this solution a soliton or "solitary wave."

The exact pulse solution for the cubic CGLE is known from Refs. [7–9]. Physically, such a solution exists as a result of two balances: the balance between the dispersion and the nonlinearity, and the balance between the amplification and the damping. As these effects depend on the pulse amplitude and width, a total balance is possible only at some fixed values of these two parameters. This is the key difference between pulses in nonconservative systems, and the soliton solutions of integrable, or, more generally, Hamiltonian systems, which usually have the amplitude as a free parameter. Unexpectedly, arbitrary-amplitude (AA) solitons exist even in nonconservative systems, and this is the subject we are discussing in this paper.

The known pulse solutions of the cubic CGLE are unstable in general. There can be two cases which depend on the range of parameters: (i) the pulse itself is unstable, or (ii) the pulse is stable but the background state surrounding it is unstable. This shows the importance of the quintic model, where, in some range of the coefficients, the pulse and the background state can be stable simultaneously. Numerical simulations of stable pulse propagation have been demonstrated in [10–12]. Regions of existence for pulselike solutions with zero velocity in the parameter space have been analyzed in [10,12,18,19]. Some analytical solutions for the pulses in the quintic model have been reported recently by several authors [12,13]. Strictly speaking, there are two stationary solutions for some sets of parameters. One of them (with the smaller amplitude) is unstable. Stability of the other one (with the larger amplitude) needs to be investigated (see [20]). However, for all solutions, the amplitude is fixed for fixed values of parameters.

In this Letter we report the analytic method which allows us to find, with the same procedure, the solitons with fixed amplitude (FA) and novel AA solitons, in both the quintic and cubic models. These AA solutions are stable for each model.

We write the CGLE in the following form (see, e.g., [2]):

$$i\psi_z + \frac{1}{2}\psi_{tt} + |\psi|^2\psi = i\delta\psi + i\beta\psi_{tt} + i\epsilon|\psi|^2\psi + i\mu|\psi|^4\psi - \nu|\psi|^4\psi, \quad (1)$$

where, for laser systems [21], z is the propagation distance, t is retarded time, $\psi(t,z)$ is the slowly varying envelope of the electrical field, δ is the linear gain at the carrier frequency, β describes spectral filtering, ϵ accounts for nonlinear amplification, μ represents a higher order correction (saturation) to the nonlinear amplification, and ν is a higher order correction term to the nonlinear refractive index.

Here we are interested in stationary solutions of Eq. (1) with zero transverse velocity, and look for a solution in the form

$$\psi(t,z) = a(t)\exp\{id\ln[a(t)] - i\omega z\}, \quad (2)$$

where $a(t)$ is a real function and d and ω are real constants. This is, obviously, a restriction because the chirp could have a more general functional dependence on t. However, this constraint allows us to find some families of solutions in analytic form. Ansatz (2) is a natural generalization of the solutions given in Refs. [7–9]. Also, the phase chirp given by (2) corresponds qualitatively to those found in numerical simulations.

It can be shown (a detailed derivation will be published elsewhere) that the ansatz (2) leads directly to the following equation for $a(t)$:

$$\frac{a'^2}{a^2} + \frac{2\nu a^4}{8\beta d - d^2 + 3} + \frac{2(2\beta - \epsilon)a^2}{3d(1 + 4\beta^2)} - \frac{\delta}{d - \beta + \beta d^2} = 0, \quad (3)$$

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where the prime stands for differentiation with respect to t,

$$d = \frac{3(1 + 2\epsilon\beta) \pm \sqrt{9(1 + 2\epsilon\beta)^2 + 8(\epsilon - 2\beta)^2}}{2(\epsilon - 2\beta)}, \tag{4}$$

and the coefficients are connected by the relation

$$\nu\left[\frac{12\epsilon\beta^2 + 4\epsilon - 2\beta}{\epsilon - 2\beta}\, d - 2\beta\right] + \mu\left[\frac{2\epsilon\beta - 16\beta^2 - 3}{\epsilon - 2\beta}\, d + 1\right] = 0. \tag{5}$$

In what follows, we consider the cubic and quintic CGLEs separately.

First we concentrate on the cubic CGLE, i.e., we put $\nu = \mu = 0$. Then Eq. (3) has a FA solution

$$a(t) = CB\,\mathrm{sech}(Bt), \tag{6}$$

where

$$B = \sqrt{\frac{\delta}{\beta d^2 + d - \beta}}, \quad C = \sqrt{\frac{3d(1 + 4\beta^2)}{2(2\beta - \epsilon)}}, \tag{7}$$

and d is given by Eq. (4) after choosing the minus sign in front of the root. The second value of d leads to an unphysical solution. The solution (6) is known as the solution of Pereira and Stenflo [8] (see also [7,9]). On the (β, ϵ) plane, the denominator in the expression for B is positive below the curve given by

$$\epsilon = \frac{\beta}{2}\, \frac{3\sqrt{1 + 4\beta^2} - 1}{2 + 9\beta^2}, \tag{8}$$

and negative above it. Note that (8) reduces to $\epsilon \approx \beta/2$ for $\beta \ll 1$, $\epsilon \approx 1/3$ for $\beta \gg 1$. It can be shown that for $\delta > 0$, the solution (6) exists, and it is stable below this curve (although the background state is unstable). For $\delta < 0$ the solution exists above the curve (8), but it is unstable (see Fig. 1).

It is easy to see that the solution (6) does not exist on the line (8). As we are moving towards this line on the (β, ϵ) plane, the soliton amplitude CB increases to infinity, and its width $1/B$ vanishes. At the same time, the amplitude-width product C remains finite. However, if we also impose the condition $\delta = 0$, a new solution, valid only on the line (8), can be found:

$$a(t) = FG\,\mathrm{sech}(Gt), \quad F^2 = d\sqrt{1 + 4\beta^2}/2\epsilon, \tag{9}$$

where G is an arbitrary positive parameter. Taking into account Eq. (8) for $\epsilon(\beta)$, we obtain the expression for d which is valid on the line (8):

$$d = \left(\sqrt{1 + 4\beta^2} - 1\right)/2\beta. \tag{10}$$

Figure 1 shows the amplitude-width product F vs β calculated along the special line (8). One can see that, for small β, the amplitude-width product is close to unity, while it follows from (8) that the chirp is small, so that the solution (9) converges to the nonlinear Schrödinger equation soliton. As β increases, the product F increases.

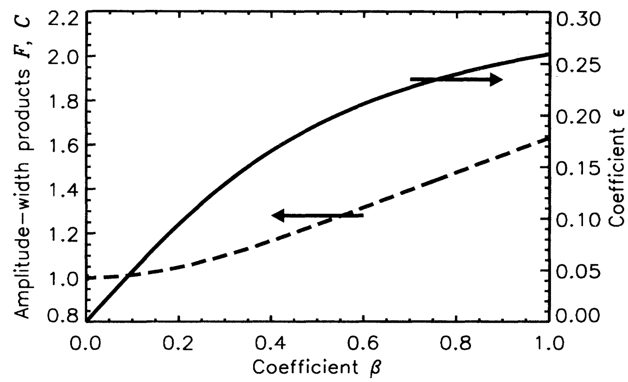

FIG. 1. Special line on the (β, ϵ) plane given by Eq. (8). This line delineates the range of existence of solition solutions with fixed amplitude. Below the line, solutions exist for $\delta > 0$ and above the line for $\delta < 0$. The arbitrary-amplitude solutions exist only on this line. The amplitude-width products $C(\beta) = F(\beta)$ (dashed line). These products are calculated on the special line (8).

The FA solitons do not exist on the special line (8). The limiting value of the amplitude-width product $C(\beta)$ for FA solitons coincide with the value $F(\beta)$ on the special line (see Fig. 1). This shows that AA solitons can be considered as limiting case of FA solitons when $\delta \to 0$. However, at each point of the special curve these solutions are a one parameter family of solutions rather than a single solution. Moreover, these new solutions have stability properties different from those for FA solitons.

The reason for the existence of the arbitrary-amplitude solitons in the cubic model is the following. When $\delta = 0$, the cubic CGLE becomes invariant under the scale transformation $\psi \to G\psi$, $t \to Gt$, $z \to G^2 z$. Hence, if we know a particular solution, the whole family can be generated using this transformation. However, some particular solution must be known. Note also that such a scale transformation is not obvious in the case of the quintic equation, which is discussed later. This problem can be studied in more detail using Lie group symmetry reductions (see, e.g., Ref. [22]).

An important problem is the stability of this solution relative to small perturbations of the initial conditions. In general, a solution is stable if, after a small perturbation, it approaches some stationary state, and is unstable if the perturbation grows exponentially. We have studied the stability of the new solutions numerically, by adding small perturbations to the exact solutions and solving the full nonlinear equation (1). The arbitrary-amplitude solutions are stable relative to both even and odd perturbations at any value of the amplitude of the pulse. Examples of stable propagation of these solitons, perturbed initially, are shown in Fig. 2. When the perturbation is symmetric, the change in the parameter F is linearly proportional to the perturbation amplitude (cf. [23]). In particular, this

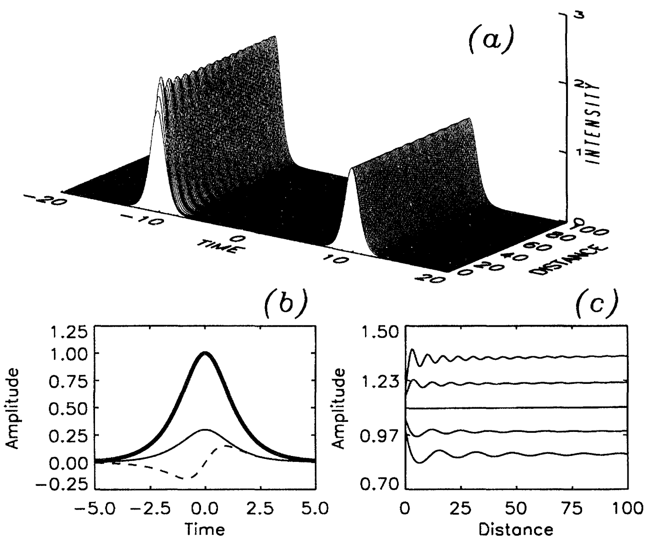

FIG. 2. (a) Stable propagation of AA solutions (9) of the cubic CGLE. Initial condition $\psi_0(t) = (1 + A_s)\,\psi_{st}(t)$ for the left soliton, and $\psi_0(t) = \psi_{st}(t) + A_s(\partial a/\partial t)$ for the right soliton, where stationary solution $\psi_{st}(t)$ is given by Eq. (9), $G = 1$, and amplitudes of perturbations $A_s = 0.1$, $A_a = 0.1$. Parameters $\delta = 0$, $\beta = 0.3$, ϵ from Eq. (8). (b) Symmetric (solid line) and antisymmetric (dashed line) perturbations of the initial conditions. Perturbations amplitudes are magnified. (c) Dynamics of the soliton amplitude for $A_s = -0.1, -0.05, 0, 0.05, 0.1$ (from top to bottom); parameters are the same as in (a).

means that, for small β, a stationary pulse can be formed from the chirp-free initial condition $\psi_0(t) = \eta\,\mathrm{sech}(\eta t)$, and, in the stationary state, $F \approx \eta$. If the perturbation is antisymmetric, F changes only in the second order. We checked numerically that the final stationary states in Fig. 2 belong to the family (9).

Now let us turn to the quintic case. The soliton solutions of the quintic CGLE exist for a wide range of values of the coefficients β, ϵ, μ, and ν. However, the ansatz (2) is the condition that restricts this range by imposing the relation (5) on them. By using the substitution $f = a^2$, we can rewrite Eq. (3) in the form

$$\frac{f'^2}{f^2} + \frac{8\nu f^2}{8\beta d - d^2 + 3} + \frac{8(2\beta - \epsilon)f}{3d(1 + 4\beta^2)} - \frac{4\delta}{d - \beta + \beta d^2} = 0\,. \quad (11)$$

Its general solution is [24]

$$f(t) = \frac{2f_1 f_2}{(f_1 + f_2) - (f_1 - f_2)\cosh(2\alpha\sqrt{f_1|f_2|}\,t)}\,, \quad (12)$$

where $\alpha = \sqrt{2|\nu/(8\beta d - d^2 + 3)|}$; f_1 and f_2 are the roots of the equation

$$\frac{2\nu f^2}{8\beta d - d^2 + 3} + \frac{2(2\beta - \epsilon)f}{3d(1 + 4\beta^2)} - \frac{\delta}{d - \beta + \beta d^2} = 0\,. \quad (13)$$

We now discuss the conditions under which the above soliton solution (12) exists. Clearly, one of the roots (we choose f_1) must be positive for the solution to exist. The second one can have either sign. If it is also positive, we choose $f_1 < f_2$. The solution given by Eq. (12) exists in two cases:

(1) $2\nu/(8\beta d - d^2 + 3) > 0$. The root f_1 is positive and the root f_2 is negative. There are no restrictions on the sign of $(2\beta - \epsilon)/d$. Hence both values of d are suitable.

(2) $2\nu/(8\beta d - d^2 + 3) < 0$. Both roots are positive. Hence $(2\beta - \epsilon)/d$ must be positive. Only a negative d satisfies this criterion.

In both cases the solution is defined by Eq. (12). However, for a given set of parameters δ, β, ϵ, and $|\nu|$, there are two solutions when ν is negative, but only one when ν is positive. The above conditions also imply that $\delta/(d - \beta + \beta d^2)$ must be always positive. This means that the restrictions on the sign of δ are the same as in the cubic case. However, for negative μ, the solution can be stable above the line (8).

The solution (12) has a singularity on the same line (8) in the plane (β, ϵ) as in the cubic case. This singularity exists when the roots f_1 and f_2 have opposite signs. If β and ϵ satisfy (8) and we have $\delta = 0$, the solution with arbitrary amplitude exists:

$$f(t) = \frac{3d(1 + 4\beta^2)P}{(2\beta - \epsilon) + S\cosh(2\sqrt{P}\,t)}\,, \quad (14)$$

where P ($P > 0$) is the parameter for the family

$$S = \sqrt{(2\beta - \epsilon)^2 + \frac{18\,d^2\nu(1 + 4\beta^2)^2}{(8\beta d - d^2 + 3)}\,P}\,. \quad (15)$$

It can be shown that, as $\mu, \nu \to 0$, the solution (14) transforms to the solution (9).

Let us analyze the conditions for this solution to exist. Taking into account that $d > 0$, $2\beta - \epsilon > 0$, and $8\beta d - d^2 + 3 > 0$, then for any $\beta > 0$ we can see from (14) and (15) that, for positive ν, the solution exists for any P, while for negative ν there is a threshold value

$$P_{\mathrm{th}} = -\frac{(2\beta - \epsilon)^2(8\beta d - d^2 + 3)}{18^2\nu(1 + 4\beta^2)}\,, \quad (16)$$

and the solution exists only for $P < P_{\mathrm{th}}$. Note that, as $\delta = 0$, β and ϵ are related by (8), μ and ν are connected by (5), so the solution (14) actually depends on two independent parameters, say, β and ν. Numerical analysis shows that the solution (14) is stable at any value of the parameter P, with respect to both even and odd amplitude perturbations (Fig. 3). The response to the perturbations is the same as for the cubic model [see Figs. 3(b) and 3(c)]. We have also studied numerically the stability of FA solitons in the quintic model. They occur to be unstable at every point of the parameter space, where exact solutions exist except of the limiting case when solitons are almost transformed into two fronts (see [20] for details).

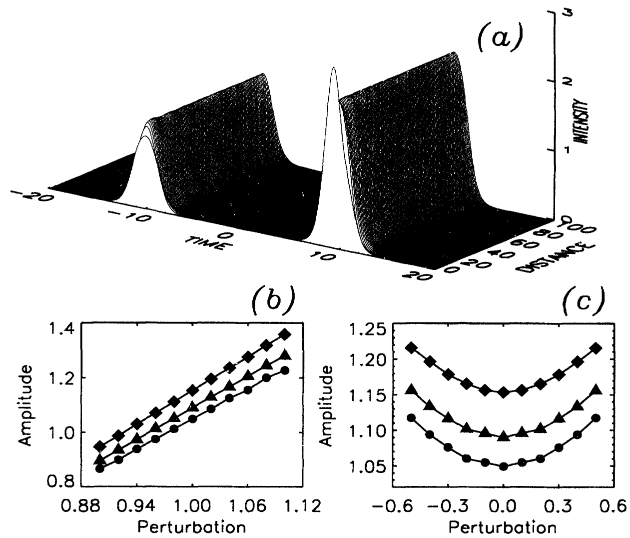

FIG. 3. (a) Stable propagation of AA solutions (2) and (15) of the quintic CGLE. Initial conditions are chosen the same way as in Fig. 2. Parameters $A_s = -0.3$, $A_a = -0.5$, $\beta = 0.3$, $\nu = -0.3$, μ and ϵ from Eqs. (5) and (8), respectively, $P = 1$. (b),(c) Soliton amplitude at $z = 100$ for $\beta = 0.1$ (circles), $\beta = 0.2$ (triangles), and $\beta = 0.3$ (diamonds), versus the amplitude of (b) symmetric and (c) antisymmetric perturbation.

The very fact that arbitrary-amplitude pulses exist is important for many applications—for instance, fiber ring lasers and optical transmission lines [2,3]. The results show that at some values of the parameters the system can be switched from the regime with hard excitation (fixed amplitude pulses) to the regime with soft excitation (arbitrary-amplitude pulses). This knowledge is important both for avoiding undesirable effects in these devices and for designing new types of all-optical switches. Note that, in the cubic model, AA solutions are the only example of stable pulses.

In conclusion, we have proposed an analytic method for seeking stationary pulse solutions of the cubic and quintic CGLE. In particular, we have discovered families of stable arbitrary-amplitude solitons for both the cubic and quintic models.

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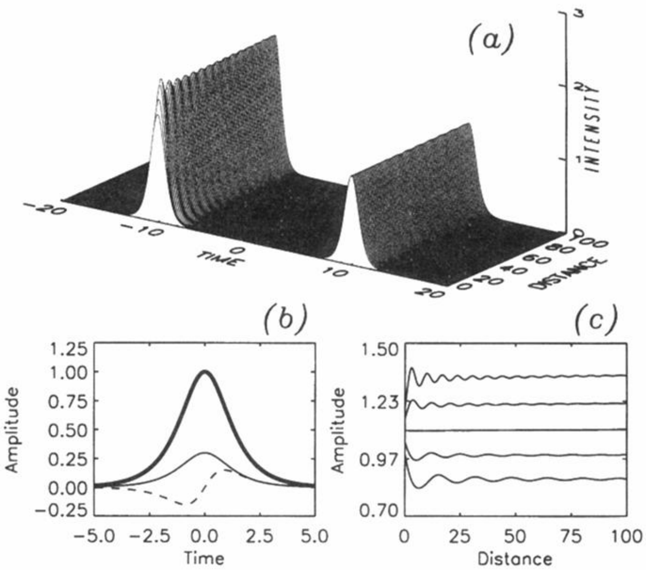

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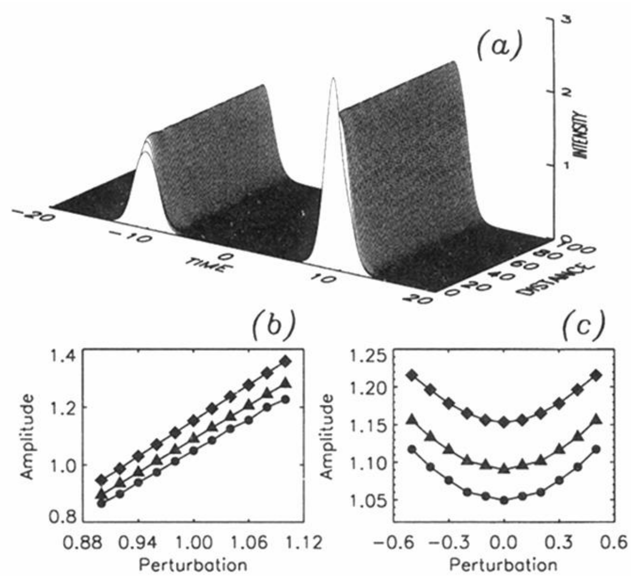

FIG. 3. (a) Stable propagation of AA solutions (2) and (15) of the quintic CGLE. Initial conditions are chosen the same way as in Fig. 2. Parameters $A_s = -0.3$, $A_a = -0.5$, $\beta = 0.3$, $\nu = -0.3$, μ and ϵ from Eqs. (5) and (8), respectively, $P = 1$. (b),(c) Soliton amplitude at $z = 100$ for $\beta = 0.1$ (circles), $\beta = 0.2$ (triangles), and $\beta = 0.3$ (diamonds), versus the amplitude of (b) symmetric and (c) antisymmetric perturbation.