Faraday Rotation and Complex-Valued Traversal Time for Classical Light Waves

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(Received 13 July 1994)

We introduce a magnetic clock approach to measure the traversal and reflection times of an electromagnetic wave through a slab, when multiple reflections are taken into account. The traversal time is a complex quantity whose real component is proportional to the Faraday rotation and whose imaginary component is proportional to the degree of ellipticity. We conclude that the complex traversal time found for an electron by several different methods is not a consequence of the quantum nature of the particle, but due to the wavelike character of the entity involved.

PACS numbers: 41.20.Jb, 42.25.–p

The question of the time spent by a particle in a given region of space has often been discussed in the literature [1–5]. There are two types of difficulties. First, the particle may have to tunnel through a barrier, in which case it has an imaginary wave vector and we do not know what is the analog of the classical velocity. Second, the final transmission amplitude is the superposition of different paths, along different trajectories or due to multiple reflections, corresponding to different traversal times. The problem has been approached from many different points of view, as shown in the recent review on the subject by Landauer and Martin [1].

The most direct method to calculate the traversal time of a particle in a barrier would be to follow the behavior of a wave packet and determine the delay due to the barrier, but this type of approach is beset with difficulties. For example, an emerging peak is not necessarily related to the incident peak in a causative way [6]. Physically more significant is the time during which a transmitted particle interacts with the barrier, as measured by some physical clock which can detect the particle's presence within the barrier. One of the principal approaches to this problem is to utilize the Larmor precession frequency of the spin, produced by a weak magnetic field acting within the barrier region, as first proposed by Baz' [7]. The amount of precession clocks the characteristic tunneling time τ_T, the so-called Büttiker-Landauer time [8,9]. Sokolovski and Baskin [10] obtained a complex traversal time, with the Feynman path-integral technique. They define a functional that measures the time spent by a Feynman path in a region and then sum this quantity over all possible paths with the weighting $e^{iS/\hbar}$, where S is the action. Which of the two components of this complex time is the most relevant depends on the experiment, but it is often the modulus of the complex time that is the magnitude directly related to the experimental measurements.

The question of the traversal time of light through a given region is equally important, but it has been seldom referred to in the literature. The advances in femtosecond technology and optoelectronics, in general, increase the inherent importance of the problem. Measurements of single photon tunneling times, using a two-photon interferometer, had to be interpreted with the existing electron theories due to the lack of a proper theory for electromagnetic waves [11]. Martin and Landauer [12] studied the problem of the traversal time of classical evanescent electromagnetic waves by following the behavior of a wave packet. Enders and Nimtz [13] found superluminal velocities in experiments on microwave transmission through undersized waveguides, corresponding to evanescent modes.

In this Letter we study the problem of the average time spent by a classical electromagnetic wave in a slab and in a layered system when we take into account the effects of multiple reflections. Analogously to the electron case, we introduce a magnetic clock that can measure the time spent by the wave packet inside the slab. We find that the time is again a complex magnitude. We conclude that this complex nature of time is due to the wavelike character of the entity involved and not a consequence of the quantum nature of the electron (or, more specifically, of the nonexistence of a Hermitian operator for time). Leavens [14] arrived at a similar conclusion from a very different point of view.

Let us first consider a slab confined to the segment $0 \leq x \leq L$ and characterized by a dielectric constant ϵ. The two semi-infinite media outside the slab are the same and are characterized by the dielectric constant ϵ_1. A linearly polarized electromagnetic plane wave enters the slab from the left at normal incidence. (We consider a plane wave for simplicity, but we will see that our results apply to any wave packet and so to any nonstationary process as should be expected in any time determination.) We take the direction of propagation as the x axis, and that of the electric field $\mathbf{E}$ in the incident wave as the z axis. A weak magnetic field $\mathbf{B}$ is applied in the x direction and confined to the slab.

The generalized principle of symmetry of the kinetic coefficients implies that $\epsilon_{ij}(\mathbf{B}) = \epsilon_{ji}(-\mathbf{B})$. The condition that absorption is absent requires that the tensor should be Hermitian $\epsilon_{ik} = \epsilon_{ki}^*$. Then the dielectric tensor

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of the slab is given by [15]

$$\epsilon_{ij} = \begin{pmatrix} \epsilon & igB \\ -igB & \epsilon \end{pmatrix}, \tag{1}$$

where g is the Faraday constant of the slab.

The linearly polarized incident wave can be represented as the sum of two circularly polarized waves with opposite directions of rotation, which then propagate through the slab with different wave vectors $k_\pm = \omega n_\pm/c$ and $n_\pm^2 = n_0^2 \pm gB$, where n_0 is the refractive index of the slab in the absence of the magnetic field, $n_0 = \sqrt{\epsilon}$.

When the wave leaves the slab it is, in general, elliptically polarized and the major axis of the ellipse is rotated with respect to the original direction of polarization. This second effect is due to the difference in phase velocity between left and right circularly polarized light and is known as the Faraday effect. The Faraday rotation θ_1 (the angle between the major axes of the ellipse and the initial direction of polarization) is proportional to the magnetic field and to the time spent by the light inside the slab. Faraday rotation is our magnetic clock and plays for light the same role as Larmor precession for electrons.

If we neglect the influence of the boundaries of the slab, which is a good approximation when $n_0 \approx n_1$, the standard Faraday rotation is

$$\theta_0 = \frac{\omega g B L}{2 c n_0} = \Omega \tau_0, \tag{2}$$

where $\Omega = \omega g B/2n_0^2$ and $\tau_0 = L n_0/c$ is the time that light with velocity c/n_0 would take to cross the slab. When reflection in the boundaries is important, the time spent by the light in the slab is in general bigger, since it can cross the slab a different number of times. Furthermore, if L is in the order of magnitude of the light wavelength in the slab, interference effects are important and may drastically change the time spent by the light in the slab.

First of all, we can relate the transmission amplitudes for right t_+ and left t_- polarized light with the Faraday angle of rotation θ_1 for any general system. This was first done by Aronov and Gasparian [16]. All the relevant information about both the angle of rotation and the degree of ellipticity is contained in the complex angle θ, defined in terms of the complex components of the electric field of the outgoing wave, E_z and E_y, as

$$\tan\theta = \frac{E_z}{E_y} = -i\,\frac{E_+ - E_-}{E_+ + E_-} = -i\,\frac{t_+ - t_-}{t_+ + t_-}. \tag{3}$$

$E_\pm$ is the electric field of the outgoing right $(+)$ and left $(-)$ polarized light, $E_\pm = E_y \pm iE_z$. As the incoming right and left polarized waves have the same amplitude, $E_\pm$ is proportional to $t_\pm$. The previous equation can be rewritten as

$$\theta = -\frac{i}{2}\ln\frac{t_+}{t_-} = \theta_1 - i\theta_2. \tag{4}$$

Let us express the complex amplitude of transmission as $t_\pm = T_\pm^{1/2}\exp\{i\psi_\pm\}$. One can easily check that the real part of the angle θ is equal to

$$\theta_1 = \frac{\psi_+ - \psi_-}{2}. \tag{5}$$

This corresponds to the Faraday rotation, which is well known to result from the phase difference between left and right polarized light. The imaginary part of θ is

$$\theta_2 = \frac{1}{4}\ln\frac{T_+}{T_-}, \tag{6}$$

and corresponds to the ratio of ellipticity.

The presence of the magnetic field produces two effects on the light: it rotates the plane of polarization and it generates ellipticity. We have seen that both effects are quantified through the complex angle θ. We can naturally associate a complex interaction time of the light in the region with the magnetic field to this complex angle, which we do next, analogously to what Büttiker did for electrons [9] (he does not consider explicitly a complex time, but his final traversal time is the modulus of the analog of our complex time).

In a small magnetic field the effective indices of refraction for the two circular polarizations are in first order in B:

$$n_\pm = n_0 \pm \frac{gB}{2n_0}. \tag{7}$$

Then θ_1 is equal to

$$\theta_1 = \frac{gB}{2n_0}\frac{\partial\psi}{\partial n_0} = \frac{\Omega n_0}{\omega}\frac{\partial\psi}{\partial n_0}, \tag{8}$$

where Ω was defined in Eq. (2), and ψ is the phase of the transmission amplitude in the absence of a magnetic field. By analogy with Eq. (2), we define the following characteristic time:

$$\tau_1 = \frac{\theta_1}{\Omega} = \frac{n_0}{\omega}\frac{\partial\psi}{\partial n_0}. \tag{9}$$

Similarly, θ_2 is given by

$$\theta_2 = \frac{\Omega n_0}{2\omega}\frac{\partial\ln T}{\partial n_0}, \tag{10}$$

where T is the transmission coefficient for $B = 0$, and we can define a second characteristic time as

$$\tau_2 = \frac{\theta_2}{\Omega} = \frac{n_0}{2\omega}\frac{\partial\ln T}{\partial n_0}. \tag{11}$$

So we have arrived at a complex characteristic interaction time τ for a classical electromagnetic wave in a given region

$$\tau = \tau_1 - i\tau_2. \tag{12}$$

The results are very similar to those obtained for a quantum particle [10,17]. The characteristic time associated with any wave (classical or quantum mechanical) has a complex magnitude. The real component of this time is analogous to the time associated with Larmor precession in the electronic case, introduced by Baz' [7]. The imaginary component is analogous to the Büttiker time associated with Zeeman splitting in the electronic case.

We believe that the Feynman path-integral technique, as used by Sokolovski and Baskin [10] for quantum particles, would produce the same result as ours when applied to electromagnetic waves.

For a slab, the transmission coefficient is given by [15]

$$T = \left\{1 + \left(\frac{n_0^2 - n_1^2}{2n_0 n_1} \sin x\right)^2\right\}^{-1}, \tag{13}$$

and the phase ψ by

$$\tan\psi = \frac{n_0^2 + n_1^2}{2n_0 n_1} \tan x, \tag{14}$$

where $x = \omega n_0 L/c$. Substituting these expressions in Eqs. (9) and (11) we obtain for the two time components

$$\tau_1 = T\left\{\frac{(n_0^2 + n_1^2)L}{2cn_1} + \frac{(n_0^2 - n_1^2)}{4n_0 n_1 \omega} \sin 2x\right\} \tag{15}$$

and

$$\tau_2 = T\frac{n_0^2 - n_1^2}{2n_0 n_1}\left\{\frac{n_0^2 + n_1^2}{2\omega n_0 n_1} \sin^2 x + \frac{L(n_0^2 - n_1^2)}{4cn_1} \sin 2x\right\}. \tag{16}$$

In some frequency ranges, the oscillatory character of the second term on the right-hand side of Eqs. (15) and (16) results in traversal times significantly smaller than the one corresponding to crossing the slab at the group velocity in the medium.

Following the analogy with the electronic case [18], we can rewrite Eqs. (15), (16), and (12) in terms of derivatives with respect to frequency as

$$\tau = -i\left[\frac{\partial \ln t}{\partial \omega} - \frac{r}{\omega}\right]. \tag{17}$$

Here r is the amplitude of reflection for the slab, given by

$$r = \exp\{i(\psi - \pi/2)\}T^{1/2}\frac{(n_0^2 - n_1^2)}{2n_0 n_1} \sin x. \tag{18}$$

The expression for the traversal time of electromagnetic waves in terms of transmission and reflection amplitudes, Eq. (17), is formally the same as for electrons traveling across any general barrier [18]. Thus we expect Eq. (17) to be of rather general validity.

From Eq. (17), or from Eqs. (15) and (16), we can easily check that the complex time satisfies $\tau(-\omega) = \tau^*(\omega)$. This implies that the real and imaginary components of time verify integral relations similar to Kramers-Kronig relations for the dielectric function [15].

The real part of the time given by Eq. (17) is related to the group velocity, as it is denoted by the derivative with respect to frequency. It produces the time that the peak of a wave packet takes to cross the slab.

Our results are valid for any wave packet provided it is longer than the width of the slab, so that interference effects due to multiple reflections are important. The traversal time of very short pulses would be the time taken to cross the slab at the group velocity, plus the two small delay times associated with the interfaces [18].

We can also consider the time spent by the reflected wave in the slab. The relevant information about both the angle of rotation and the degree of ellipticity of the reflected wave is now contained in the complex angle $\theta^{(R)}$ given by

$$\theta^{(R)} = -\frac{i}{2} \ln\frac{r_+}{r_-}. \tag{19}$$

In analogy with transmission, we associate two times with the two components of this complex angle, and we find that the real component of time is equal to τ_1, while the imaginary component is equal to $-\tau_2|r|^2/T$. So we arrive at the following expression for the reflection time $\tau^{(R)}$:

$$\tau^{(R)} = \tau_1 + i\frac{|r|^2}{T}\tau_2. \tag{20}$$

The same relationship holds in the electronic case [9].

We extended our calculations to a layered system, with the help of the method developed by Aronov and Gasparian [16], and found that Eq. (17) is still valid, within the analog of the effective mass approximation for electrons. As for electrons [19], τ_1 corresponds to the density of states and τ_2 to the localization length of electromagnetic waves.

We have applied Eq. (17) to the microwave experiment of Enders and Nimtz [13] which measured the time taken by an electromagnetic wave packet to traverse an evanescent waveguide region and obtained superluminal speeds. They calculated the time by Fourier transforming the experimental frequency data to the time domain, so that they do not take into account the r/ω contribution in Eq. (17). The strong dependence of the transmission coefficient with frequency in the evanescent region causes a shift in the peak of the frequency distribution, which, combined with the fact that waveguides are highly dispersive systems, produces two interesting effects. First, the transmitted packet travels at a higher speed than the incident one, the same as happens with electrons tunneling through a barrier [1]. Second, there is a correction to the delay time (i.e., the real component τ_1) that can be written in terms of the imaginary component τ_2 of the complex time. For an incident Gaussian wave packet of width β, in the angular frequency domain, this correction is given by:

$$\tau_1^{(\mathrm{eff})} = \tau_1 + \beta^2\tau_2. \tag{21}$$

This is another example of how the two time components give us the full information of the traversal time problem.

In Fig. 1 we plot the transmission velocity obtained from Eq. (17) (dashed line) and take into account the previous correction (solid line) as a function of the thickness of the evanescent region together with the experimental data (points) of Enders and Nimtz [13]. The

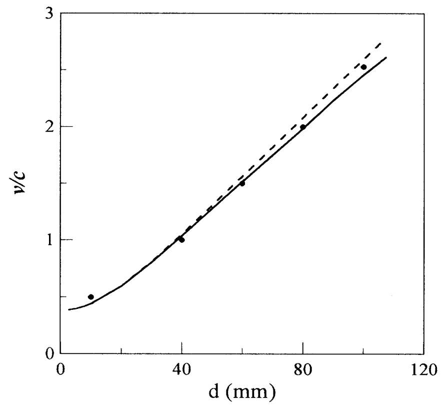

FIG. 1. Transmission velocity obtained from Eq. (17) (dashed line) and from Eq. (21) (solid line) as a function of the thickness of the evanescent region together with the experimental data (points) of Enders and Nimtz [13].

horizontal line corresponds to the vacuum speed of light. Our theoretical results fit fairly well the experimental points.

We have introduced a magnetic clock to measure the time in which a light wave interacts with the magnetic field while in the slab. This time is related to the Faraday rotation and the ellipticity of the polarization of the outgoing light. The method is similar to the Larmor precession approach for electrons, although some of the problems appearing in the electronic case are absent. In particular, the experimental measurement of the Faraday rotation and of the ellipticity are much easier to perform than the observation of the spin rotation.

Alternatively, we could use an electric field to measure the interaction time of the electromagnetic wave in a given region. The Kerr effect, for a transverse electric field, and the Pockels effect, for a longitudinal field, are potential electric clocks for light waves.

We would like to thank C. R. Leavens and M. Pollak for helpful discussions and a critical reading of the manuscript, and to acknowledge D. Sokolovski for useful conversations and the Dirección General de Investigación Científica y Técnica, Project No. PB 93/1125 and sabbatical support for V. G., and the European Economic Community, Contract No. SSC*-CT90-0020, for financial support.

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