Correlators of Spectral Determinants in Quantum Chaos

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We generalize an approach recently introduced to study arbitrary correlators of spectral determinants in quantum chaotic systems with broken T invariance. The utility of this method for obtaining generating functions for a variety of universal correlators is illustrated with several applications.

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Despite the success of the supersymmetry approach [1] in describing the universal properties that characterize the phenomena of quantum chaos, its application is still at present restricted to a subset of correlators that involve at most two points. More traditional methods of random matrix theory [2] provide a complementary approach, although their application again brings some restrictions. In this Letter we will generalize a third approach, originally introduced by Guhr [3], to study a whole class of universal correlation functions applicable to quantum chaotic systems with broken T invariance (unitary symmetry). The virtue of this approach, which relies on a superalgebraic construction, lies in its straightforward application. Since, in contrast to Efetov's supersymmetry method, final expressions are not presented in terms of an integration over a restricted saddle-point manifold (the nonlinear σ model), this technique draws no distinction between two- and higher-point functions. All are equally tractable.

The function that we consider involves the general many-point correlation of spectral determinants

$$W(\{U;V\}) = a_{lm}\left\langle \frac{\prod_{i=1}^{m}\det(V_i - H)}{\prod_{j=1}^{l}\det(U_j - H)}\right\rangle, \qquad (1)$$

where the normalization $a_{lm} = \langle|\det H|^{l-m}\rangle$ ensures that W remains finite as the number of levels $N \longrightarrow \infty$. We will restrict attention to even values of $l + m$ where the correlation function becomes universal on scales of $\{U\}$ and $\{V\}$ comparable to the average level spacing. In the same limit the correlator for $l + m$ odd is vanishingly small.

Apart from its role as a generating function for density of states (DOS) fluctuations [3], W is related to a number of distribution functions. One example involves the curvature distribution derived in Ref. [4]. A straightforward generalization leads to the following expression for the generating function of the joint curvature distribution:

$$K(s,\omega) = \sum_{\mu}\left\langle \exp\left[-\frac{is}{2}\frac{\partial^2\epsilon_\mu}{\partial x^2}\right]\delta(\epsilon_\mu)\nu(\omega)\right\rangle = \frac{\pi^2}{3}\frac{\omega^3}{\omega - is}W(is;0,0,0,\omega,\omega), \qquad (2)$$

where $\nu(\epsilon) = \mathrm{Tr}\,\delta(\epsilon - H/\Delta)$ denotes the dimensionless operator for DOS. Here we have introduced $\epsilon_\mu = E_\mu/\Delta$ with $\Delta = \langle E_{\mu+1} - E_\mu\rangle$, and $x = X\sqrt{C(0)}$ which parametrizes an arbitrary external perturbation with $C(0) = \langle(\partial\epsilon_\mu/\partial X)^2\rangle$ [5,6]. A second example concerns the generating function for the joint distribution of local DOS

$$L(s,\omega) = \left\langle \exp\left[2is\gamma\sum_{\mu}\frac{|\psi_\mu(\eta)|^2}{\epsilon_\mu^2 + \gamma^2}\right]\nu(\omega)\right\rangle = \frac{\omega^2 + \gamma^2}{\omega^2 + \alpha^2}W(i\alpha,-i\alpha;i\gamma,-i\gamma,\omega,\omega), \qquad (3)$$

where $\alpha^2 = \gamma^2 - is\gamma$, $\mathrm{Re}\,\alpha > 0$, and $\psi_\mu(\eta)$ denotes the μth wave function at site η. Equations (2) and (3) are both straightforwardly obtained by exploiting the statistical independence of the spectra and wave functions and making use of the Porter-Thomas distribution [7].

Without the additional correlation to the DOS, both have been examined in the recent literature. The curvature distribution has been used as the first indicator of universality in statistics that depend on an external parameter [4,5,8,9], while the distribution of local DOS, measured through NMR, has been studied as a signature of chaotic behavior in mesoscopic metallic grains [10,11].

Although Eq. (1) seems amenable to traditional orthogonal polynomial methods of random matrix theory [2] such as that used by Ref. [11], the theory rapidly becomes intractable as the number of points in the correlator increases. At the same time, as will become clear, the disparity of the order of the determinants in the numerator and denominator rules out conventional supersymmetry approaches. We will show that for unitary symmetry a third approach, which generalizes a method introduced by Guhr [3], yields the following exact expression involving the dimensionless parameters $u_i = U_i/\Delta$ and $v_i = V_i/\Delta$,

$$W(\{U;V\}) = A_N \sum_{P[\{v\}]} F_{lm}(\{u;v\}) \times e^{i\pi\left(\sum_{i=1}^{l_*} u_i - \sum_{i=l_*+1}^{l} u_j - \sum_{r=1}^{m_*} v_r + \sum_{s=m_*+1}^{m} v_s\right)}, \qquad (4a)$$

$$F_{lm}(\{u;v\}) = \frac{\prod_{r=1}^{m_*}\prod_{j=l_*+1}^{l}(u_j - v_r)\prod_{i=1}^{l_*}\prod_{s=m_*+1}^{m}(u_i - v_s)}{\prod_{i=1}^{l_*}\prod_{j=l_*+1}^{l}(u_i - u_j)\prod_{r=1}^{m_*}\prod_{s=m_*+1}^{m}(v_r - v_s)}. \qquad (4b)$$

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A_N denotes a normalization constant, and, for convenience, we have made the ordering such that $\{u_1, \ldots, u_{l_*}\}$ represent the l_* parameters for which $\mathrm{Im}\, u_i > 0$ while $\{u_{l_*+1}, \ldots, u_l\}$ denote those elements with $\mathrm{Im}\, u_i < 0$. With $m_* = l_* + (m - l)/2$, the summation is performed over the ${}^mC_{m_*} = m!/m_*!(m - m_*)!$ permutations which interchange elements v_r and v_s between the two summations in the exponential. Our goal will be to obtain this expression using a Gaussian distribution of Hermitian random matrices with unitary symmetry, and to demonstrate its utility by obtaining explicit expressions for $K(s, \omega)$ and $L(s, \omega)$. A more detailed discussion of this general approach together with some applications can be found in a longer paper [12]. The coincidence of the statistical properties of random matrix ensembles with the universal properties of quantum chaos is well studied in the literature (see, for example, Ref. [13]), and we will not discuss it further here.

The starting point, which is common to the supersymmetry method, involves constructing an expression for W in the form of a Gaussian integral,

$$W(\{U; V\}) = a_{lm}(-1)^{Nl_*}(2\pi i)^{N(m-l)} \int d[\psi]\langle e^{-S_0 - S_1}\rangle,$$
$$S_0 = -i\psi^\dagger g \hat{Z}\psi, \qquad S_1 = i\psi^\dagger g H \psi, \tag{5}$$

where $\hat{Z} = \mathrm{diag}(U_1, \ldots, U_l, V_1, \ldots, V_m)$, and $\psi^T = (\vec{S}_1, \ldots, \vec{S}_l, \vec{\chi}_1, \ldots, \vec{\chi}_m)$ denotes the $(l + m) \times N$ component fields with complex bosonic $\vec{S}$ and fermionic $\vec{\chi}$ variables [14]. Formally the convergence of Eq. (5) is assured by the inclusion of the metric $g = \mathrm{diag}(\mathbb{1}_{l_*}, -\mathbb{1}_{m+l-l_*})$ [15].

The ensemble average over the Gaussian distribution of H

$$P(H)dH = C_N \exp\left[-\frac{\pi^2}{2N\Delta^2}\,\mathrm{Tr}H^2\right]dH, \tag{6}$$

where C_N is the normalization constant [2], leads to an effective action,

$$\langle e^{-S_1}\rangle = e^{-S_{\mathrm{eff}}},$$
$$S_{\mathrm{eff}} = -\frac{N\Delta^2}{2\pi^2}\,\mathrm{STr}\left[i\sum_\mu g^{1/2}\psi_\mu \otimes \psi_\mu^\dagger g^{1/2}\right]^2. \tag{7}$$

The trace or supertrace operation for supermatrices follows the convention $\mathrm{STr}M = \mathrm{Tr}M_{BB} - \mathrm{Tr}M_{FF}$, where M_{BB} and M_{FF}, respectively, denote the boson-boson and fermion-fermion block of M. The quartic interaction of the fields can be decoupled by the Hubbard-Stratonovich transformation

$$e^{-S_{\mathrm{eff}}} = A_{lm}\int d[Q]\exp\left[-\frac{N}{2}\,\mathrm{STr}Q^2 + \frac{iN\Delta}{\pi}\,\psi^\dagger g^{1/2}Qg^{1/2}\psi\right], \tag{8}$$

where Q denote $(l + m) \times (l + m)$ supermatrices with a block structure reflecting that of $\psi \otimes \psi^\dagger$, and $A_{lm} = 2^{-(l+m)/2}(N/\pi)^{(l-m)^2/2}$. Combining Eq. (8) with Eq. (5), integrating over ψ, and shifting the integration variables $Q \longrightarrow Q - \pi\hat{z}/N$, we obtain

$$W(\{U; V\}) = a_{lm}A_{lm}\int d[Q]\exp\left[-\frac{N}{2}\,\mathrm{STr}\left(Q - \frac{\pi\hat{z}}{N}\right)^2 - N\mathrm{STr}\ln\left(\frac{N\Delta}{\pi}\,Q\right)\right], \tag{9}$$

where $\hat{z} = \hat{Z}/\Delta$.

Thus far, the method departs from the conventional supersymmetry approach [1,15,16] only in that it allows for a different number of bosonic and fermionic variables. At this stage, however, instead of forming the usual expansion around the saddle point of Eq. (9) to obtain a nonlinear σ model, we will exploit the fact that the expression is of the form of an Itzykson-Zuber integral [17] over the full pseudounitary supergroup that diagonalizes the supermatrices Q. This approach was introduced by Guhr [3] who used an equal number of bosonic and fermionic variables to examine the high-point correlator of DOS fluctuations in unitary ensembles (see also Ref. [18]).

The interaction S_{eff} in Eq. (7) is invariant under the action of the pseudounitary supergroup $\mathrm{SU}(l_*, l - l_*/m)$ [15]. This is reflected in the structure of the Hubbard-Stratonovich field Q. The supermatrix Q can be diagonalized by a matrix $T \in \mathrm{SU}(l_*, l - l_*/m)$ such that $Q = T^{-1}ST$, where $S = \mathrm{diag}(b_1, \ldots, b_l, if_1, \ldots, if_m)$ denotes the matrix of eigenvalues with b_i and f_i taking values on the range $-\infty$ to ∞. The integration measure is then given by

$$d[Q] = \mathrm{const} \times B_{lm}^2(S)\,d[S]\,d\mu[T], \tag{10a}$$

$$B_{lm}(S) = \frac{\prod_{r<s}^{l}(b_r - b_s)\prod_{p<q}^{m}(if_p - if_q)}{\prod_{i=1}^{l}\prod_{j=1}^{m}(b_i - if_j)}. \tag{10b}$$

Previous studies have demonstrated the extension of the Itzykson-Zuber integral to the superunitary group $\mathrm{SU}(l/m)$ [19,20]. Similar considerations suggest a further extension to the pseudounitary supergroup $\mathrm{SU}(l_*, l - l_*/m)$. However, since these arguments are somewhat technically involved we will reserve their discussion to a longer paper [12] and make use of several applications of Eq. (4) to justify the validity of this approach. The result of the psuedounitary integration closely parallels that of the superunitary group and leads to the expression [19,20]

$$\int d[T] e^{\pi \mathrm{STr}(T^{-1}ST\hat{z})} = \text{const} \times \frac{\det(e^{\pi b_i u_j})_{1\le i,j\le l^*} \det(e^{\pi b_i u_j})_{l_*+1\le i,j\le l} \det(e^{-i\pi f_i v_j})_{1\le i,j\le m}}{B_{lm}(S) B_{lm}(\pi\hat{z})}. \tag{11}$$

The determinants involve only those permutations of eigenvalues S which can be obtained from one another by the action of the psuedounitary transformation $T \in \mathrm{SU}(l_*, l - l_*/m)$. The expression that appears from the combination of Eqs. (9), (10), and (11) involves the sum of many terms arising from the expansion of the determinants. However, the interchange of integration variables in S shows all contributions in the expansion to be identical. Taking one contribution, and shifting back $S \to S + \pi\hat{z}/N$, we obtain

$$W(\{U;V\}) = \text{const} \times \int d[S] \frac{B_{lm}(S + \pi\hat{z}/N)}{B_{lm}(\hat{z})} \exp\left\{-\frac{N}{2}\mathrm{STr}S^2 - N\mathrm{STr}\ln\left[\frac{N\Delta}{\pi}\left(S + \frac{\pi\hat{z}}{N}\right)\right]\right\}. \tag{12}$$

The problem of evaluating the correlator of spectral determinants has been reduced to a set of $l + m$ real integrations over the eigenvalues of S. It is at this stage that we make use of large N to estimate the integral by means of the saddle-point approximation as described in Ref. [3]. For problems of physical interest the dimensionless source $\hat{z}$ is of order unity and does not affect the saddle points: $S_0 = \mathrm{diag}(b_{01}, \ldots, b_{0l}, if_{01}, \ldots, if_{0m})$ where the elements $\{b_{0i}\}$ and $\{if_{0i}\}$, in principle, take values of $\pm i$. However, the saddle-point values of the bosonic variables $b_{0i} = \pm i$ lie off the real axis, and a deformation of the integration contour is required to reach them. This has to be done in such a way that singularities of the integrand are not crossed. In particular, the signs of the eigenvalues, $\mathrm{Im}b_{i0}$ must be chosen to be consistent with those of $\mathrm{Im}u_i$. This implies that $[S_0]_{BB} = ig_{BB}$. Conversely, the saddle points associated with the fermionic degrees of freedom f_{0i} lie on the real axis, and must all be taken into account.

The leading order contribution to the integral comes from the value of the integrand at the saddle points. Fluctuations give corrections which are as small as $1/N$ [3]. As a result, we obtain

$$W(\{U;V\}) = \text{const} \times (-1)^{N(m_* - l_* + (l-m)/2)} \sum_{\{S_0\}} \frac{B_{lm}(S_0 + \pi\hat{z}/N)}{B_{lm}(\hat{z})} e^{i\pi\left(\sum_{i=1}^{l_*} u_i - \sum_{i=l_*+1}^{l} u_j - \sum_{r=1}^{m_*} v_r + \sum_{s=m_*+1}^{m} v_s\right)}, \tag{13}$$

where m_* is the number of $+i$'s in the fermionic block of S_0, and $\sum_{\{S_0\}}$ denotes the sum over all possible saddle points. This involves the interchange of all possible signs of if_{0i} in the fermionic sector of S_0. However, although the number of such terms is 2^m, not all of them are of the same order. Some are as small as $1/N$, significantly reducing the number of terms that must be taken into account. This can be seen by considering a typical factor arising from the numerator of the integrand: $if_{0i} + \pi y_i/N - if_{0j} - \pi y_j/N$. If $if_{0i} = if_{0j}$, this factor is proportional to $1/N$, while if $if_{0i} = -if_{0j}$, y_i and y_j can be neglected and it becomes of order unity. If there are l_* terms which take the value of i in the bosonic sector of S_0, and m_* in the fermionic sector, there is a relative factor of $N^{(m-l-m_*+l_*)(m_*-l_*)}$ multiplying the contribution of this point to Eq. (13). The maximum of this factor is achieved when $m_* = l_* + (m - l)/2$. Then only ${}^mC_{m_*}$ saddle points have to be taken into account. Applying this condition we arrive at the expression shown in Eq. (4).

Equation (4) represents the central result of this Letter. As a generating function, the correlator of spectral determinants allows access to a number of useful correlation functions. To conclude, we will apply Eq. (13) to determine algebraic expressions for several examples. As a simple application we begin by considering the generating function for local DOS

$$P(s) = \left\langle \exp\left[2is\gamma \sum_\mu \frac{|\psi_\mu(\eta)|^2}{\epsilon_\mu^2 + \gamma^2}\right]\right\rangle = W(i\alpha, -i\alpha; i\gamma, -i\gamma), \tag{14}$$

where the notation is taken from Eq. (3). As a supersymmetric combination, this average can be compared with the known result first obtained by Efetov and Prigodin [10]. Adopting the approach above, $l_* = 1$, $m_* = 1$, and we need consider saddle-point contributions from ${}^2C_1 = 2$ terms: $S_0 = \mathrm{diag}(i, -i; i, -i)$ and $S_0 = \mathrm{diag}(i, -i; -i, i)$. Applying Eq. (4) we obtain

$$P(s) = P_1(\alpha, \gamma),$$

$$P_\mu(\alpha, \gamma) = \sum_{\sigma=\pm 1} \frac{(\alpha + \sigma\gamma)^2}{4\sigma^\mu \alpha\gamma} e^{-2\pi(\alpha - \sigma\gamma)}. \tag{15}$$

This result coincides with that obtained in Ref. [10] (see also Ref. [11]).

Having verified this approach with a known example, let us consider the two generating functions defined previously in Eqs. (2) and (3). Both cases require the application of a nonsupersymmetric construction. Beginning with the joint curvature distribution, applying Eq. (4) to the case where $s > 0$, we obtain

$$K(s,\omega) = \frac{\omega e^{-\pi s}}{\omega - is}\left\{\frac{(is-\omega)^2}{\omega^2}R_2(\omega) - \frac{is}{\omega^3}\left[\omega\left(\frac{1}{\pi}+s\right) - \frac{2is}{\pi}\right]e^{-\pi i\omega}\sin(\pi\omega) + \frac{s}{\omega^2}[\omega(i+\pi\omega)+s(2-i\pi\omega)]\right\}, \quad (16)$$

where $R_2(\omega) = 1 - \sin^2(\pi\omega)/(\pi\omega)^2$ denotes the two-point correlator of DOS [2]. [An expression for values of $s < 0$ can be found by complex conjugation of Eq. (16).]

The validity of Eq. (16) can be tested by considering two limiting cases. First, for $s = 0$, $K(0,\omega)$ describes the two-point correlator of DOS fluctuations [2]. Second, as $\omega \to \infty$, the generating function should collapse to the disconnected average involving the average DOS and the Fourier transform of the known curvature distribution [4,9]. An inspection confirms that both limits are realized by Eq. (16). We remark that, in the limit of $\omega \to 0$, $K(s,\omega) = R_2(\omega)\omega/(\omega - is)$. Its Fourier transform implies a joint curvature distribution which vanishes for $\omega\partial^2\epsilon_\mu/\partial x^2 > 0$ and decays exponentially at a rate proportional to ω for $\omega\partial^2\epsilon_\mu/\partial x^2 < 0$. This contrasts with the power law decay of the uncorrelated curvature distribution ($\omega \to \infty$).

Turning to the second generating function, Eq. (4) implies

$$L(s,\omega) = i\pi\omega\frac{\gamma^2-\alpha^2}{\gamma^2+\omega^2}P_2(\alpha,i\omega) + \frac{1}{\pi}\frac{\pi\omega^2+\pi\alpha^2+\alpha}{\gamma^2+\omega^2}P_1(\alpha,\gamma) - \frac{\gamma}{\pi}\frac{\omega^2+\alpha^2}{(\gamma^2+\omega^2)}P_2(\alpha,\gamma). \quad (17)$$

where $P_{1,2}$ are defined in Eq. (15). In this case, the validity of Eq. (17) can also be checked in the same limits. For $s = 0$ ($\alpha = \gamma$), $L(0,\omega)$ corresponds to the average DOS, which is independent of ω. On the other hand, as $\omega \to \infty$ the decoupling of the average recovers the generating function for the local DOS already obtained in Eq. (15). Again, both limits bare inspection.

Equations (16) and (17) represent just two examples of where the average W can be exploited. Further examples include generalizations of the distribution of resonance conductance peaks in quantum dots [21] as well as the sensitivity of chaotic wave-function intensities to changes in an external perturbation [22]. The extension of this approach to orthogonal and symplectic symmetry relies on the construction of the appropriate Itzykson-Zuber integral corresponding to Eq. (11). However, to our knowledge, for the pseudo-orthogonal and pseudosymplectic supergroup, such a generalization has yet to be found.

To conclude, in this Letter we have obtained an exact analytical expression for a whole class of correlators that characterize quantum chaos for systems without T invariance. The utility of this approach has been demonstrated with the derivation of two distribution functions.

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