Maximally Supersymmetric String Theories in $D < 10$

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The existence of maximally supersymmetric solutions to heterotic string theory that are not toroidal compactifications of the ten-dimensional superstring is established. We construct an exact fermionic realization of an $N = 1$ supersymmetric string theory in $D = 8$ with non-simply-laced gauge group Sp(20). Toroidal compactification to six and four dimensions gives maximally extended supersymmetric theories with reduced rank (4, 12) and (6, 14), respectively.

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Finiteness is a robust property of the perturbative amplitudes of the known superstring theories. $N = 4$ supersymmetric Yang-Mills theory is known to be finite in four dimensions [1], and there is growing evidence that the theory exhibits an extension of Olive-Montonen strong-weak coupling duality known as S duality [2,3]. A generalization of the Olive-Montonen duality of $N = 4$ theories has also been identified in $N = 1$ supersymmetric Yang-Mills theory [4]. In string theory, conjectures for S duality have mostly been explored in the context of toroidal compactifications of the ten-dimensional heterotic string to spacetime dimensions $D < 10$ [5].

It would be helpful to have insight into the generic moduli space, and the generic duality group, of such *maximally supersymmetric* string theories. We will therefore consider the possibility of exact solutions to string theory beyond those obtained by dimensional reduction from a ten-dimensional superstring. These solutions are exact in the sigma model (α') expansion but are perturbative in the string coupling constant. To be specific, we will construct solutions to heterotic string theory, i.e., with $(N_R, N_L) = (2, 0)$ superconformal invariance on the world sheet. Our construction is, however, quite general, and the conclusions can be adapted to solutions of any closed string theory in any spacetime dimension.

Toroidal compactification of the ten-dimensional $N = 1$ heterotic string to six (four) dimensions results in a low-energy effective $N = 2$ ($N = 4$) supergravity coupled to 20 (22) Abelian vector multiplets, giving a total of 24 (28) Abelian vector gauge fields with gauge group $(\mathrm{U}(1))^{24}$ ($(\mathrm{U}(1))^{28}$), respectively. Four (six) of these Abelian multiplets are contained within the $N = 2$ ($N = 4$) supergravity multiplets. At enhanced symmetry points in the moduli space the Abelian group $(\mathrm{U}(1))^{20}$ ($(\mathrm{U}(1))^{22}$) is enlarged to a simply laced group of rank 20 (22). The low-energy field theory limit of such a solution has maximally extended spacetime supersymmetry. Since all of the elementary scalars appear in the adjoint representation of the gauge group, symmetry breaking via the Higgs mechanism is adequate in describing the moduli space of vacua with a *fixed* number of Abelian multiplets.

In this Letter we show that there exist maximally supersymmetric vacua with four-, six-, and eight-dimensional Lorentz invariance that are not obtained by toroidal compactification of a ten-dimensional heterotic string. The total number of Abelian vector multiplets in the four-dimensional theory can be reduced to just *six*, namely, those contained within the $N = 4$ supersymmetry algebra. This is consistent with known theorems on the world-sheet realizations of extended spacetime supersymmetry in string theory [6]. In the world-sheet description of an $N = 4$ supersymmetric solution of the heterotic string in four dimensions, the internal right moving superconformal field theory of central charge $c_R = 9$ is required to be composed of nine free bosons. A reduction of the rank of the low-energy gauge group in an $N = 4$ solution implies that the internal left-moving conformal field theory of central charge $c_L = 22$ is not entirely composed of free bosons. This is unlike the 4D toroidal compactifications described by Narain [7] where *both* right- and left-moving conformal field theories are free boson theories.

As a consequence, it will also be possible to realize non-simply-laced gauge symmetry consistent with the maximally extended supersymmetries. We will construct such solutions using real fermionization [8]. Exact solutions to heterotic string theory obtained in this construction are examples of rational (2,0) superconformal field theories, where the underlying chiral algebras also have a world-sheet fermionic realization. In order to have an unambiguous identification of the vertex operator algebra in the fermionic construction, it is essential to have explicit knowledge of the correlators of the real fermion conformal field theories [9].

Eliminating longitudinal and timelike modes, the number of transverse degrees of freedom describing a vacuum with D-dimensional Lorentz invariance is $(c_R, c_L) = (\frac{3}{2}(D - 2), (D - 2) + (c_R^{int}, c_L^{int}))$. In this class of exact solutions, the internal degrees of freedom have an *equiva-*

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lent world-sheet fermionic realization with $(3(10 - D), 2(26 - D))$ Majorana-Weyl fermions. The world-sheet fermionic realization is convenient, both as an explicit calculational tool and because it allows us to construct, consistent with finiteness and anomaly cancellation, an exact solution to string theory, which embeds *a specified low-energy matter content*.

We restrict ourselves to fermionic realizations where the world-sheet fermions are Majorana-Weyl, with periodic or antiperiodic boundary conditions only. All of the right-moving world-sheet fermions will be paired into Weyl fermions, or equivalently free bosons, as required by the extended spacetime supersymmetry. A free boson conformal field theory implies, with no loss of generality, the existence of an Abelian current in the right-moving superconformal field theory. In maximally supersymmetric solutions the allowed right-moving chiral algebras are, therefore, restricted to level one simply laced affine Lie algebras [6,7]. This follows from the fact that for a Lie algebra with roots of equal length the central charge of the level one realization also equals the rank of the algebra, i.e., the number of Abelian currents.

A free fermionic realization with n Weyl (complex) fermions exists for any of the following affine Lie algebras: $SO(2n)$, $U(n)$, and E_8 (for $n = 8$), in addition to the Abelian algebra $(U(1))^n$. In toroidal compactifications that have an equivalent free fermionic realization these properties also extend to the allowed left-moving chiral algebras and, hence, to the observed non-Abelian gauge symmetry in these solutions.

Incorporating *real* fermion world-sheet fields in the left-moving internal conformal field theory will enable us to construct maximally supersymmetric solutions that embed non-simply-laced gauge symmetry, i.e., gauge groups with roots of unequal length. Such solutions necessarily lie in a moduli space where the gauge group has rank <28. This is evident from the formula for the central charge of an affine Lie algebra,

$$c = \frac{k\,\mathrm{Dim}(G)}{k + \tilde{h}}\,, \tag{1}$$

where the dual Coxeter numbers $\tilde{h}$ of the non-simply-laced algebras $SO(2n + 1)$, $Sp(2n)$, G_2, and F_4 are, respectively, $2n - 1$, $n + 1$, 4, and 9. Note that the dimensions of the dual algebras $SO(2n + 1)$ and $Sp(2n)$ are identical, given by $\mathrm{Dim}(G) = n(2n + 1)$. However, unlike the simply laced algebras, the central charge does not equal the rank of the group even at level $k = 1$, and does not, in fact, coincide for the algebra and its dual. Real fermion realizations exist for all of the non-simply-laced affine algebras. Extending a world-sheet fermionic realization of the generators of the affine algebra to a $(2,0)$ superconformal field theory that is an exact solution to heterotic string theory, however, requires consistency with modular invariance of the one-loop vacuum amplitude and with world-sheet supersymmetry [9]. These conditions can be quite restrictive and, in fact, preclude $N = 1$ supersymmetric solutions in ten spacetime dimensions with non-simply-laced gauge symmetry.

Now consider the possibility of non-simply-laced gauge symmetry in $D < 10$. For example, an affine realization of the rank ten algebra $Sp(20)$ at level one requires central charge $c = \frac{35}{2}$. Appending a single real fermion with $c = \frac{1}{2}$ gives $c = 18$, making this a plausible candidate for the gauge group of an $N = 1$ spacetime supersymmetric solution in $D = 8$. It is not difficult to verify the existence of such a solution using its fermionic realization.

We will adopt the notation of [8,9]. The tree level spectrum is described by the one-loop vacuum amplitude, which sums over sectors labeled by the associated spin structure of the world-sheet fermions. The $N = 1$ spacetime supersymmetry charges are embedded in the spin structure of eight right-moving Majorana-Weyl fermions, which we will label ψ_μ, $\mu = 1, \ldots, 6$, ψ_7, and ψ_{10}. The spin-$\frac{3}{2}$ generator of the $(1,0)$ world-sheet supersymmetry is the operator

$$T_F(\bar{z}) = i\sum_{\mu=1}^{6} \psi_\mu \partial_{\bar{z}} X^\mu + i \sum_{k=2,3} \psi_{3k+1}\psi_{3k+2}\psi_{3k+3}\,. \tag{2}$$

The first six right movers therefore carry a (transverse) spacetime index. In sectors contributing spacetime bosonic and fermionic components of an $N = 1$ supermultiplet, these eight fermions are, respectively, Neveu-Schwarz and Ramond. In particular, the *untwisted* sector $\mathcal{U}$, in which all of the world-sheet fermions are Neveu-Schwarz, contributes the bosonic components of the $N = 1$ supergravity multiplet in eight dimensions. It also contributes two massless Abelian multiplets, each associated with an internal right-moving Weyl fermion: $\psi_8 + i\psi_{11}$ and $\psi_9 + i\psi_{12}$. Thus the full gauge group of this model is $Sp(20) \times (U(1))^2$. Note that the Ramond vacuum of the right-moving fermions ψ_7, ψ_{10} is constrained by modular invariance to be aligned with that of the first six right movers. Thus, there cannot exist modular invariant solutions to heterotic string theory with extended spacetime supersymmetry in $D = 8$, as expected from the viewpoint of the low-energy effective Lagrangian.

The remaining massless spectrum is arranged into $D = 8$ $N = 1$ Yang-Mills supermultiplets, each containing 6 spacetime vector components, 8 spinor components, and 2 scalar components [10]. The sectorwise decomposition of the 210 states in the adjoint representation of $Sp(20)$ is most easily described by the regular embedding

$$Sp(20) \supset (SO(5))^5 \supset (SO(4))^5 \sim (SU(2))^{10}. \tag{3}$$

The untwisted sector $\mathcal{U}$ contributes states corresponding to all 30 long roots, and a subset (20) of the short roots of $Sp(20)$. These states transform, respectively, in the adjoint (10 copies of a **3**) and the spinor (10 copies of a doublet) representation of its $(SU(2))^{10}$ subgroup. The states are identified by fermionic charge: the roots and weights of the rank ten subgroup are embedded in the fermionic charge of ten *Weyl* fermions. In the fermionic construction these are obtained by pairing 20 Majorana-Weyl left movers, $\psi_{2l+1}(z) + i\psi_{2l+2}(z) = \lambda_l(z)$, $l = 0,\ldots,9$.

The remaining 16 left-moving Majorana-Weyl fermions are *real* fermions. The vertex operator construction for an SO$(2n+1)$ algebra requires a single real fermion, in addition to n Weyl fermions. The long-root lattice of SO$(2n+1)$ coincides with the root lattice of SO$(2n)$, $\mathbf{\Lambda}_{\mathrm{L}}(B_n) = D_n$. Thus the $n(2n-1)$ Majorana-Weyl fermion bilinears are the currents corresponding to long roots, while those corresponding to the short roots are the $2n$ bilinears containing the single real fermion. In this example, of the $\frac{20\times19}{2}$ Neveu-Schwarz fermion bilinear currents contributed by the untwisted sector only $\frac{5\times5\times4}{2}$ remain after Gliozzi-Scherk-Olive projection from four *twisted* sectors, $\mathcal{T}_1,\ldots,\mathcal{T}_4$, in which some of the fermions are Ramond. The untwisted sector therefore contributes a total of 400 states: the eight bosonic components of an $N=1$ supermultiplet transforming in the adjoint representation of the non-simply-laced group $(\mathrm{SO}(5))^5$.

Extension of this vertex operator construction to a symplectic current algebra requires conformal dimension $(h_R, h_L) = (0,1)$ operators corresponding to the additional short roots. These are contributed by the twisted sectors. The currents are composite operators constructed out of sixteen *twist* fields, i.e., dimension $(0,\frac{1}{16})$ operators in the Majorana-Weyl fermion field theory.

The twisted sectors $\mathcal{T}_i$ were chosen so as to generate the necessary projection on the untwisted sector. They will simultaneously determine the internal right-moving chiral algebra: in this solution, the four internal right-moving fermions, ψ_8, ψ_9, ψ_{11}, and ψ_{12}, are either all Neveu-Schwarz or all Ramond in every sector of the Hilbert space. Thus the underlying right-moving chiral algebra is SO(4). Possible twists are, of course, subject to constraints from modular invariance and world-sheet supersymmetry. Given a set of valid $\mathcal{T}_i$, modular invariance of the one-loop vacuum amplitude automatically generates additional twisted sectors in the Hilbert space. Thus, in this example, the $\mathcal{T}_i + \mathcal{T}_j$, $i \neq j$, also contribute massless states in the spectrum. Each of the ten twisted sectors contributes 128 states: 8 bosonic components of an $N=1$ supermultiplet transforming in the 16-dimensional spinor representation of an $(\mathrm{SU}(2))^4$ subgroup. Sp(20) has ten distinct $(\mathrm{SU}(2))^4$ subgroups, each corresponding to a different twisted sector. Combining the 400 untwisted sector states with these 1280 states gives all 8×210 bosonic components of an $N=1$ supermultiplet transforming in the adjoint representation of Sp(20).

It is straightforward to construct the twisted sector vertex operator corresponding to a given weight. We will use the bosonic realization for the corresponding free field vertex operator. A state transforming as a spinor weight α of $(\mathrm{SU}(2))^4$ corresponds to a dimension $(0,\frac{1}{2})$ operator, $j_{\mathrm{free}}(z)$, obtained by bosonization,

$$\lambda_l^\dagger\lambda_l \to \partial\phi_l, \qquad j_{\mathrm{free}}(z) = \hat{C}(\alpha)e^{i\alpha\cdot\phi}, \tag{4}$$

where $\alpha\cdot\alpha = 1$, $l = 0,\ldots,9$, and the $\hat{C}(\alpha)$ are suitable cocycle operators. This free field vertex operator must be dressed by four pseudo-Weyl fermion spin fields, $\sigma_l^{\pm}$, $l = 1,\ldots,4$, so as to give a current. These spin fields are identified by pseudocomplexifying, i.e., pairing, the real fermions in a twisted sector [9]. Thus

$$J_{ijkl}(z) = j_{\mathrm{free}}(z)\,(\sigma_i^+\sigma_j^+\sigma_k^+\sigma_l^+ + \sigma_i^-\sigma_j^-\sigma_k^-\sigma_l^-), \tag{5}$$

where $i \neq j \neq k \neq l$, giving a dimension-$(0,1)$ twisted sector current. Verification of the vertex operator algebra for Sp(20) is now straightforward.

This completes the discussion of the massless spectrum of the $N=1$ supersymmetric Sp(20) heterotic string in eight dimensions. Anomaly cancellation is particularly simple in this theory: there is no gravitational anomaly in $D=8$ dimensions [11], the right-moving U(1)'s are nonanomalous, and Sp(20) is an anomaly free gauge group. Compactification on a torus will give anomaly free theories with maximally extended supersymmetry in lower dimensions. Quite generally, compactification on an (n,n)-dimensional D_n lattice has an equivalent fermionic realization in terms of (n,n) Weyl fermions. Recall that the fermionic description of the Sp(20) string theory contained an extra left-moving real fermion. Appending this real fermion to the Weyl fermion realization of the SO$(2n)$ current algebra extends it to a realization of SO$(2n+1)$. It is straightforward to verify, as we have done, the existence of an $N=2$ Sp(20) $\times$ SO(5) solution in six dimensions and an $N=4$ Sp(20) $\times$ SO(9) solution in four dimensions with fermionic realizations. Thus toroidal compactification of the eight-dimensional $N=1$ Sp(20) heterotic string gives an $N=4$ theory in four dimensions with only *twenty* Abelian vector multiplets, or rank (6, 14), at generic points in the moduli space. The target space duality group clearly has an $\mathrm{O}(4,4;Z)\backslash\mathrm{O}(4,4)/(\mathrm{O}(4)\times\mathrm{O}(4))$ subgroup corresponding to the moduli space of the torus, but it will be extended by the background modes of the $D=8$ supergravity Yang-Mills theory [7,12].

The moduli spaces of six-dimensional solutions are of particular interest in exploring string-string duality. The conjectured S duality of the heterotic string compactified on a six-dimensional torus has been shown to follow as a consequence of target space T duality of the type IIA string theory: compactified on the $K3$ surface, this string theory is dual to the heterotic string compactified on a four-dimensional torus [13–15]. As described above, toroidal compactification of the ten-dimensional $E_8\times E_8$ string and the eight-dimensional Sp(20) string gives rank 24 and rank 16 moduli spaces, respectively. Twisting the Sp(20) $\times$ SO(5) solution gives a solution with exceptional gauge symmetry: $F_4\times F_4\times$ Sp(8). It is likely that these solutions belong to the same moduli space. We have also constructed a new family of $N = 2$ solutions with 12 Abelian multiplets at generic points in the moduli space. This moduli space contains enhanced symmetry points with higher level realizations of the gauge symmetry: $SU(9)_2$, $(SU(5)_2)^2$, and $(SU(3)_2)^4$. Twisting the $(SU(5)_2)^2$ solution gives a solution with the orthogonal gauge group $(SO(9)_2)^2$.

We have constructed fermionic realizations of a large range of four-dimensional $N = 4$ and six-dimensional $N = 2$ supersymmetric solutions to the heterotic string with semi-simple groups of varying rank, containing both simply laced and non-simply-laced factors, and with part or all of the gauge symmetry realized at a higher level. It should be stressed that four-dimensional $N = 4$ supersymmetry need not always arise via toroidal compactification from a higher dimensional theory. The clearest evidence for this is the existence of an $N = 4$ four-dimensional solution where the gauge symmetry is reduced to the minimum consistent with the world-sheet supersymmetry constraints. Its fermionic realization uses a spin structure block of 44 left-moving real fermions. The number of Abelian vector multiplets in this $N = 4$ theory is just *six*.

The development of fermionization techniques [8,9] has enabled the efficient sampling of new classes of exact solutions to string theory. It is important to focus on those aspects of the solutions that have generic implications for our understanding of string theory. We would like to stress that there exist additional maximally supersymmetric solutions, in any space-time dimension, which do *not* have fermionic realizations. A simple example in $D = 4$ is toroidal compactification with gauge group $(SU(3))^3 \times E_8 \times E_8$. The particular choices of affine Lie group, rank, or Kac-Moody level, obtained in the fermionic construction should not be emphasized. On the other hand, the existence of maximally supersymmetric theories with distinct target space duality groups, and the fact that non-simply-laced and simply laced gauge groups enter on an equal footing, are generic observations relevant for further study.

In conclusion, we note that the construction of alternative four-dimensional $N = 4$ string theories is a useful step towards constructing simpler pedagogical models for studying the low-energy physics of string theory. Any four-dimensional $N = 1$ supersymmetric solution to heterotic string theory inherits some of its structure from a parent $N = 4$ solution. Four-dimensional $N = 4$ heterotic string theories of lower rank are a more appealing starting point for constructing pedagogical $N = 1$ models via twisting, because the models will inherit a reduced massless particle spectrum. In toroidal compactifications the massive modes of the string spectrum were completely determined by the low-energy symmetries: extended supergravity and gauge symmetry. This is no longer true in the maximally supersymmetric solutions constructed in this paper. The mechanism by which finiteness is achieved in these solutions does not rest wholly upon the finiteness of the low-energy field theory limit. Twisting such solutions to construct pedagogical $N = 1$ models with chiral matter will teach us about new, and intrinsically stringy, mechanisms for achieving finiteness.

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