Spatial Structure and Field-Line Diffusion in Transverse Magnetic Turbulence

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We examine magnetic surfaces and randomization of field lines with fluctuations transverse to a uniform magnetic field. Analogy with passive scalar transport in inviscid 2D flow provides realizations of magnetic surfaces and motivates a nonperturbative statistical approach. The stochastic wandering of magnetic field lines leads to diffusive perpendicular transport. For two-component fluctuations, appropriate for solar wind turbulence, the diffusion coefficient is a nonadditive combination of slab and 2D coefficients, approaching the latter in the small amplitude limit.

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Understanding of the effects of space and astrophysical plasma turbulence has advanced greatly through the use of idealized models of turbulent fluctuations. Plasma heating [1] and transport of turbulence [2] are particularly sensitive to the nature of these fluctuation models, as are the propagation [3,4] and acceleration [5] of cosmic rays. The most common assumptions have been that the fluctuations admit either isotropic or slab symmetry, the former associated with classical turbulence theory and the latter with linear plasma wave theory. Recently, there has been considerable interest in models with more general rotational symmetries, motivated by simulations [6,7], magnetohydrodynamic (MHD) theory [8,9], and direct observations [10,11]. These models have had a remarkable effect on various space [2,12] and astrophysical [13] applications, in some cases greatly improving correspondence between theoretical predictions and observations [9,12,14]. However, little appears to have been done to describe these fluctuations either intuitively or in terms of the mathematics of field-line random walk. In this Letter we examine a family of transverse fluctuation models, focusing on a description of the wandering and diffusion of magnetic surfaces, i.e., flux tubes and field lines. These properties are contrasted to those of the slab model, the standard in space and astrophysics for nearly thirty years.

We find that magnetic surfaces experience a rapid shredding and tangling in composite transverse models. Thus we see a failure of the common assumption that flux tubes, in general, remain identifiable as they tangle uniformly about a constant guide field. To quantify this property of the magnetic surfaces, we calculate the (coarse-grained) diffusion coefficient for field-line random walk, adapting methods developed for fluid and guiding center plasma diffusion (see [15–17] and references therein). Surprisingly, we find that the small fluctuation limit of the composite model is the 2D transport result and not the slab result [4,18,19].

Field lines and flux tubes are examples of magnetic surfaces, that is, smooth surfaces in space everywhere tangent to the local magnetic field $\mathbf{B}$. There is obviously a large latitude in constructing examples. A simple way to construct a family of contiguous magnetic surfaces is using a scalar function α that satisfies $\mathbf{B}\cdot\nabla\alpha = 0$. Each value of α then specifies a distinct surface; i.e., the surface $\alpha = \alpha_0$ is defined by the set of points $\{\mathbf{x} \mid \alpha(\mathbf{x}) = \alpha_0\}$. The usual practice is to speak of each disjoint part as a separate magnetic surface.

We are concerned with statistically axisymmetric models of magnetic fluctuations $\mathbf{b}$ that are transverse to a uniform mean field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. The total magnetic field is $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}(x,y,z)$, and $\mathbf{b}\cdot\mathbf{B} = 0$. This category includes slab, 2D, quasi-2D, and two-component models [10] that include both 2D and slab contributions. In terms of a vector potential $A_z(\mathbf{x})\hat{\mathbf{z}}$, the magnetic field is

$$\mathbf{B} = B_0\hat{\mathbf{z}} + \nabla A_z \times \hat{\mathbf{z}}\,. \tag{1}$$

Note that the surfaces of A_z are magnetic surfaces only when A_z is independent of z.

For transverse fluctuations, magnetic surfaces satisfy

$$\frac{\partial \alpha}{\partial z} + \frac{\mathbf{b}}{B_0}\cdot\nabla\alpha = 0\,. \tag{2}$$

Now consider incompressible 2D flow with fluid velocity $\mathbf{v}$, in which a passive scalar ψ evolves according to

$$\frac{\partial \psi}{\partial t} + \mathbf{v}\cdot\nabla\psi = 0\,. \tag{3}$$

These two equations are identical under the replacements $\alpha \to \psi$, $t \to z$, and $\mathbf{v} \to \mathbf{b}/B_0$. This gives us a convenient method for generating magnetic surfaces in a given realization of transverse turbulence.

We generate turbulent magnetic fields by specifying a Fourier representation of A_z and weighting each wavenumber component according to the desired spectral dependence in the model. For most of the remaining discussion, we separate explicitly the slab component of fluctuations $\mathbf{b}^{\mathrm{slab}}(\mathbf{z})$. In addition, assume that $A_z = A_z(\mathbf{x}_\perp) = A_z(x,y)$ only, giving rise to 2D fluctuations. For this composite (or two-component) model, the fluctuation is

$$\mathbf{b} = \mathbf{b}^{\mathrm{2D}}(x,y) + \mathbf{b}^{\mathrm{slab}}(z)\,, \tag{4}$$

where $\mathbf{b}^{\mathrm{2D}}(x,y) = \nabla A_z(x,y) \times \hat{\mathbf{z}}$. We shall denote by δb, δb_{slab}, and δb_{2D} the root mean square amplitude of

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the total fluctuation, and the slab and 2D components, respectively. The specified modal spectra produce reduced spectra that are fairly flat at low k and exhibit a power law proportional to $k^{-5/3}$ at high k. The fluctuations have $\delta b = B_0/3$ and Gaussian statistics. For a specified $\alpha(z = 0)$, magnetic surfaces are computed from Eq. (2) using an adaptation of a Fourier spectral method 2D hydrodynamic and passive scalar code.

Figures 1 and 2 illustrate examples of magnetic surfaces calculated in this way. In both figures the left plane is a perspective view of the arbitrary starting function on the $z = 0$ plane, and horizontal extension depicts evolution along the mean field. The spatial structure of the magnetic surfaces can be discerned by examining their intersections with the slices shown in the simulation cube, where they appear as curves of constant α.

Figure 1 represents a composite model with 80% 2D and 20% slab components of turbulence, this admixture representing a reasonable model for solar wind turbulence [12,14]. The boundary value of α on the $z = 0$ plane was chosen to be $A_z(z = 0)$. This choice is special. A_z describes only the 2D fluctuations and is not a function of z; surfaces of constant A_z are unvarying cylinders extended along z. Without the slab component, this particular choice of surfaces of α would have the same regular property. In contrast, here the evolution of α is quite involved, and the resemblance to turbulent mixing and diffusion in two-dimensional hydrodynamics is obvious.

In Fig. 2 the starting function is chosen without regard for A_z, a more general circumstance. We have specified $\alpha(z = 0) \sim \sin(k_x x)\sin(k_y y)$, with $k_x = k_y$ just below k_0, the break in the turbulent spectrum. The top frame shows magnetic surfaces for purely slab turbulence. Because $\mathbf{b}(z)$ is the same for all field lines, their wandering resembles a collection of random walks that occur in lock step. There is no variation in the pattern of α. It is simply displaced without distortion.

The middle frame depicts magnetic surfaces for purely 2D turbulence. Unlike the slab case, individual field lines follow independent paths. The magnetic surfaces fold and produce finer and finer structure with increased z, just as an initially smooth passive scalar would become convoluted and mixed in time. The bottom frame shows

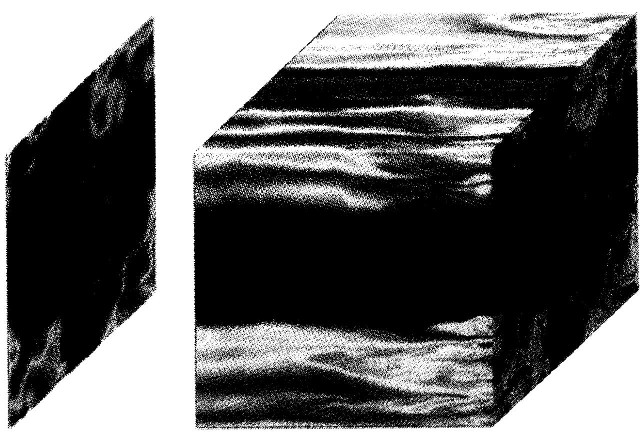

FIG. 1(color). Magnetic surfaces for a composite model of 80% 2D and 20% slab MHD turbulence. The left plane shows $\alpha(x, y, z = 0) = A_z(z = 0)$, the numerical box is a 128^3 grid, and colors represent values of the function α.

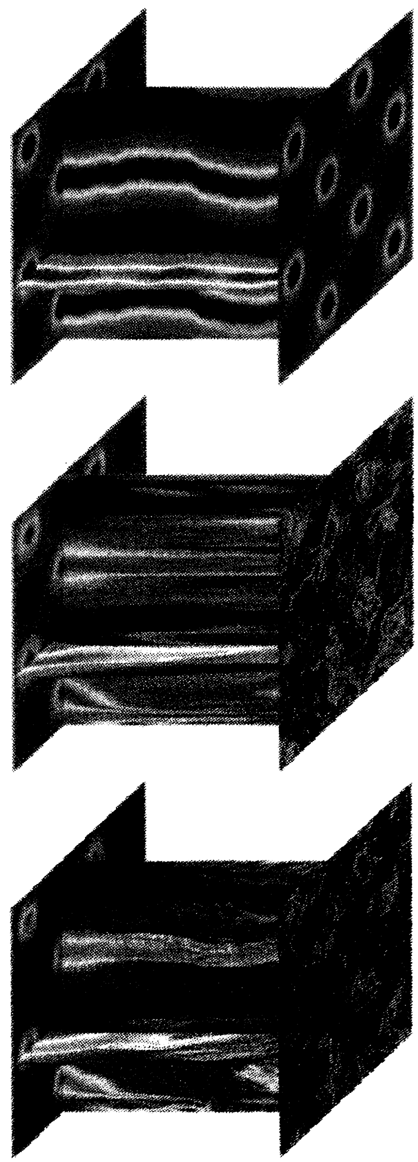

FIG. 2(color). Magnetic surfaces for top, purely slab turbulence; middle, purely 2D turbulence; and bottom, composite of 80% 2D and 20% slab turbulence. All cases have $\alpha(x, y, z = 0) \sim \sin k_x x \sin k_y y$, with $k_x = k_y \lesssim k_0$ the turnover in the turbulent spectrum.

the effect of adding slab to 2D turbulence—the surfaces become convoluted more quickly.

The wandering of magnetic field lines depends on the statistics of $\mathbf{b}$ and is expected to give rise to turbulent diffusion in analogy with hydrodynamics. [Eqs. (2) and (3) are ideal, but the coarse-grained variables [20] are expected to exhibit diffusion]. In the passive scalar problem, the eddy diffusion coefficient is expected to be of the order of $u\lambda$, where u is the typical turbulent velocity and $\lambda \approx$ the correlation length. According to the analogy between magnetic surfaces and Eq. (3), we expect the effective field-line diffusion coefficient to be $D_\perp \approx \tilde{\lambda}\delta b/B_0$, where $\tilde{\lambda}$ is some appropriate large length scale.

The diffusion coefficient $D_\perp$ for α can be defined to describe the randomization of α in the (x, y) plane as z changes, corresponding to the diffusion coefficient for perpendicular wandering of magnetic field lines [4,18,19]. For self-diffusion of fluid elements, it is found by integrating the Lagrangian correlation over the full trajectory [16,17]. Here, the displacements are proportional to the components of $\mathbf{B}$, from which

$$D_\perp = \frac{\langle(\Delta \mathbf{x}_\perp)^2\rangle}{4\Delta z} = \frac{1}{2}\int_0^\infty dz\, \frac{\langle \mathbf{b}(\mathbf{x}_\perp(z), z)\cdot\mathbf{b}(0,0)\rangle}{B_0^2}. \tag{5}$$

The angle brackets denote an ensemble average, and the statistics are assumed to be homogeneous. Jokipii and Parker [4,18,19], in effect, started with this expression but neglected dependence of the integrand upon $\mathbf{x}_\perp$ when $\delta b \ll B_0$, reducing it to the Eulerian form. Their result is exact for the slab model.

We consider diffusion for the slab and 2D composite model given by Eq. (4). Because slab and 2D fluctuations are uncorrelated (orthogonal under ensemble averaging), the integrand in (5) can be separated additively into slab and 2D contributions. However, field-line statistics appear on the right hand side of Eq. (5), so the net diffusion coefficient will not simply be the sum of two independent terms.

The slab term follows immediately from the definition of the parallel correlation length, λ_z and $\delta b_{\rm slab}$,

$$\int_0^\infty dz\, \frac{\langle \mathbf{b}^{\rm slab}(z)\cdot\mathbf{b}^{\rm slab}(0)\rangle}{2B_0^2} = \frac{\delta b_{\rm slab}^2 \lambda_z}{2B_0^2} \equiv D_{\rm slab}. \tag{6}$$

This is Jokipii and Parker's result [4,18,19].

To calculate the contribution from 2D fluctuations, we employ key elements of the approach used by Montgomery and co-workers [15,17]. First, express the relevant terms in Eq. (5) in a Fourier representation in the perpendicular coordinate,

$$\langle \mathbf{b}^{\rm 2D}(\mathbf{x}_\perp(z))\cdot\mathbf{b}^{\rm 2D}(0)\rangle = \sum_{\mathbf{k}_\perp}\langle|\mathbf{b}^{\rm 2D}(\mathbf{k}_\perp)|^2 e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp(z)}\rangle. \tag{7}$$

Note that the manipulations are in the region of infinite extent in z but enclosed by a finite box in the (x, y) directions. The amplitude $|\mathbf{b}^{\rm 2D}(\mathbf{k}_\perp)|^2$ of the 2D fluctuations, with wave number $\mathbf{k}_\perp$ satisfying $\mathbf{k}_\perp\cdot\hat{\mathbf{z}} = 0$, would become the spectrum after ensemble averaging. The statistics of the magnetic fluctuations can be separated from those of the individual trajectories by using Corrsin's hypothesis [17,21], which implies

$$\langle|\mathbf{b}_\perp^{\rm 2D}(\mathbf{k}_\perp)|^2 e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp(z)}\rangle = \langle|\mathbf{b}_\perp^{\rm 2D}(\mathbf{k}_\perp)|^2\rangle\langle e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp(z)}\rangle. \tag{8}$$

This separation reflects the fact that the random trajectory $\mathbf{x}_\perp(z) = (x(z), y(z))$ is highly irregular and sensitive to phases of the magnetic fluctuations, and not just the spectrum. A second key step [17] is to assume that $x(z)$ and $y(z)$ are identically distributed but uncorrelated Gaussian random variables, from which it follows by expansion of the exponential and the definition of the diffusion coefficient that

$$\langle e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp(z)}\rangle = e^{-k_\perp^2 D_\perp z}. \tag{9}$$

Integrating over z gives the two-dimensional contribution to the diffusion coefficient

$$\frac{1}{2}\sum_{\mathbf{k}_\perp}\frac{\langle|\mathbf{b}^{\rm 2D}(\mathbf{k}_\perp)|^2\rangle}{B_0^2}\int_0^\infty e^{-k_\perp^2 D_\perp z}\,dz = \frac{D_{\rm 2D}^2}{D_\perp}. \tag{10}$$

The quantity

$$D_{\rm 2D} = \left(\sum_{\mathbf{k}_\perp}\frac{\langle|\mathbf{b}^{\rm 2D}(\mathbf{k}_\perp)|^2\rangle}{2k_\perp^2 B_0^2}\right)^{1/2} \tag{11}$$

is, by virtue of the analogy between (2) and (3), essentially identical to the diffusion coefficient for case (i) in Ref. [17], the coefficient of self-diffusion in "frozen-in" 2D turbulence. In the present case, if the slab component vanishes, the perpendicular field-line diffusion coefficient would be $D_\perp = D_{\rm 2D}$.

For the composite model, we assemble the contributions from (6) and (10), and find the relation $D_\perp = D_{\rm 2D}^2/D_\perp + D_{\rm slab}$, the solution of which is

$$D_\perp = \frac{D_{\rm slab} + \sqrt{D_{\rm slab}^2 + 4D_{\rm 2D}^2}}{2}. \tag{12}$$

Note that we can write the 2D diffusion coefficient at $D_{\rm 2D} = \delta b_{\rm 2D}\tilde{\lambda}/B_0$, where $\tilde{\lambda}$ is now defined by the $k_\perp$ weighting of the spectrum in Eq. (11). This should be compared to the slab result, $D_{\rm slab} = \delta b_{\rm slab}^2\lambda_z/2B_0^2$. Suppose that we fix the ratio of slab to 2D fluctuation energy, as well as the ratio of λ_z to $\tilde{\lambda}$. The immediate conclusion is that, in the limit of small fluctuation amplitude, $\delta b/B_0 \to 0$ and $D_\perp \to D_{\rm 2D}$. Thus, for the two component model, the 2D diffusion result is the proper small amplitude limit.

The general case of transverse fluctuations can now be treated. We begin again with Eq. (5) but now pass to the limit of an infinite domain in all three coordinates. For a large box of side L, the field-line diffusion coefficient becomes

$$D_\perp = \lim_{L\to\infty}\int_0^L dz \left\langle \sum_{\mathbf{k}} \frac{|\mathbf{b}(\mathbf{k})|^2}{2B_0^2}\, e^{i[\mathbf{k}_\perp\cdot\mathbf{x}_\perp(z) + k_z z]} \right\rangle. \tag{13}$$

In parallel to the development above, we employ Corrsin's hypothesis [17,21]. In this case we pass to the limit $L \longrightarrow \infty$, adopt a Fourier integral representation, and designate the magnetic spectral density of the fluctuations as $S(\mathbf{k})$, where $\delta b^2 = \int d^3k\, S(\mathbf{k})$. Again, assuming that the random displacements $\mathbf{x}_\perp(z)$ are described by a diffusion process with Gaussian statistics; Eq. (13) then reduces to

$$D_\perp = \int d^3k \frac{S(\mathbf{k})}{2B_0^2} \int_0^\infty e^{-k_\perp^2 D_\perp z + ik_z z}\, dz\,. \qquad (14)$$

Integration in z yields an integral equation

$$1 = \int d^3k \frac{S(\mathbf{k})}{2B_0^2} \frac{k_\perp^2}{(D_\perp k_\perp^2)^2 + k_z^2}\,, \qquad (15)$$

which determines the diffusion coefficient $D_\perp$ for transverse magnetic fluctuations with arbitrary spectral distribution. It shares with other formal expressions for diffusion coefficients [such as Eq. (11), see also [16,17]] a sensitivity to power at extremely long wavelengths because of the inverse weighting in wave number. It is also straightforward to show that the solution to Eq. (15) for the slab and 2D composite model reduces to the expression in Eq. (12).

In conclusion, we find that transverse fluctuation models that are more general than the slab model produce magnetic surfaces that tangle and shred along the guide field direction. The orderly weaving of identifiable flux tubes about the mean field, a familiar conclusion from the slab model, does not occur in general. The spatial behavior of transverse fluctuations is analogous to the ideal turbulent diffusion of a passive tracer in 2D incompressible hydrodynamics. Motivated by this analogy, we derive a perpendicular field-line diffusion coefficient by employing statistical methods [17] that involve no expansion in small parameters. The result for a slab and 2D two-component model, such as is appropriate to the solar wind, is a diffusion coefficient very different from the standard slab result [4,18,19], even in the small amplitude limit. Further generalizations of these methods are expected to have consequences for scattering and transport of high energy charged particles in astrophysical plasmas.

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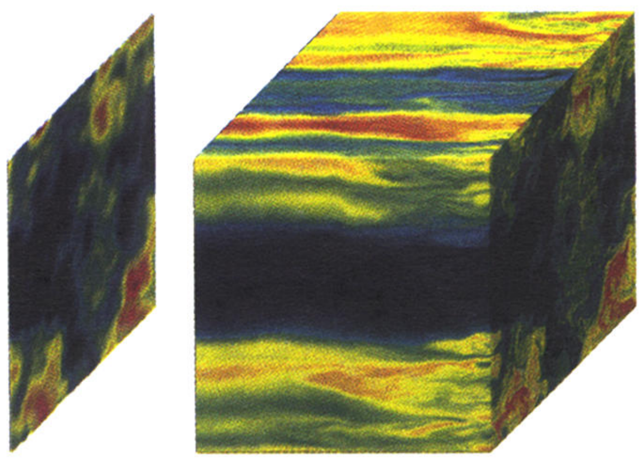
[1] J. V. Hollweg, J. Geophys. Res. **91**, 4111 (1986).
[2] S. Oughton and W. Matthaeus, J. Geophys. Res. (to be published).
[3] J. Jokipii and E. N. Parker, Astrophys. J. **155**, 799 (1969).
[4] J. Jokipii and E. N. Parker, Astrophys. J. **155**, 777 (1969).
[5] P. Legage and C. Cesarsky, Astron. Astrophys. **118**, 223 (1983).
[6] J. V. Shebalin, W. H. Matthaeus, and D. Montgomery, J. Plasma Phys. **29**, 525 (1983).
[7] S. Oughton, E. Priest, and W. Matthaeus, J. Fluid Mech. **280**, 95 (1994).
[8] D. Montgomery, Phys. Scr. **T2/1**, 83 (1982).
[9] G. Zank and W. Matthaeus, J. Geophys. Res. **97**, 17 189 (1992).
[10] W. H. Matthaeus, M. L. Goldstein, and D. A. Roberts, J. Geophys. Res. **95**, 20 673 (1990).
[11] K. Desai, C. Gwinn, and P. Diamond, Nature (London) **372**, 754 (1994).
[12] J. Bieber *et al.*, Astrophys. J. **420**, 294 (1994).
[13] J. R. Jokipii and J. Giacalone, Geophys. Res. Lett. **95**, 20 673 (1995).
[14] J. Bieber, W. Wanner, and W. Matthaeus (to be published).
[15] D. Montgomery, in *Plasma Physics, 1972 Les Houches Summer School of Theoretical Physics,* edited by C. DeWitt and J. Peyraud (Gordon and Breach, New York, 1973).
[16] J. B. Taylor and B. McNamara, Phys. Fluids **14**, 1492 (1971).
[17] Y. Salu and D. C. Montgomery, Phys. Fluids **20**, 1 (1977).
[18] J. R. Jokipii, Astrophys. J. **146**, 480 (1966).
[19] J. R. Jokipii and E. Parker, Phys. Rev. Lett. **21**, 44 (1968).
[20] U. Frisch, in *Lecture Notes on Turbulence,* edited by J. R. Herring and J. C. McWilliams (World Scientific, Teaneck, NJ, 1989), p. 234.
[21] S. Corrsin, in *Atmospheric Diffusion and Air Pollution, Advances in Geophysics, Vol. 6,* edited by F. Frenkiel and P. Sheppard (Academic, New York, 1959), p. 161.



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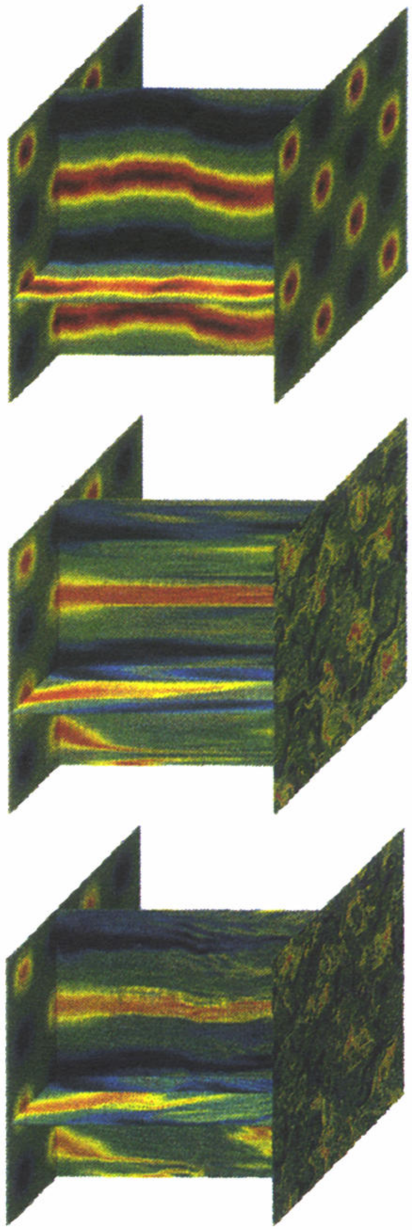

FIG. 2(color). Magnetic surfaces for top, purely slab turbulence; middle, purely 2D turbulence; and bottom, composite of 80% 2D and 20% slab turbulence. All cases have $\alpha(x, y, z = 0) \sim \sin k_x x \sin k_y y$, with $k_x = k_y \lesssim k_0$ the turnover in the turbulent spectrum.