Detecting Periodic Unstable Points in Noisy Chaotic and Limit Cycle Attractors with Applications to Biology

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Recently, biological preparations which are thought to be chaotic have been controlled using algorithms based on the detection and manipulation of periodic unstable points. The dynamics of these systems are, however, contaminated with noise; thus detection becomes a statistical process. Here we show that low dimensional chaos can be reliably detected with large noise contamination and distinguished from noisy limit cycles. We also examine a purely chaotic high dimensional system.

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Attempts to experimentally detect low dimensional chaotic behavior in biological preparations have a long history. Such efforts are usually designed to yield a measure, for example, the dimension or entropy, of the underlying dynamics by applying some algorithm to an experimentally obtained time series [1]. Attention has been drawn to a wide variety of systems ranging from the dynamics of heart beats [2] to brain function [3]. However, because the most commonly used algorithms and their close variants require long data sets [4], are susceptible to noise contamination, or are unreliable when applied to relatively high dimensional systems [5], efforts to detect low-dimensional chaos in the aforementioned systems, which are nearly always found to be noisy and/or high dimensional and nonstationary, have been questioned [6]. In spite of these difficulties, careful statistical studies of large data sets have recently produced evidence for low dimensional behavior in spinal cord reflexes and hippocampal slices [7]. However, if the detection of chaos is ever to become a useful analytical tool applied to biological systems, it will be necessary to detect it reliably in the presence of noise using relatively short data sets comparable to those from current biological experiments.

The cyclic theory of chaos [8] offers a different approach. In this picture, an infinite set of periodic unstable points (PUP's) forms a "skeleton" upon which the chaotic dynamics is built. This means that orbits will, with some probability, visit the PUP's, approaching along a stable manifold and departing along an unstable manifold. These encounters, even though they may be rare, can be detected by identifying a suitable characteristic, for example, by measuring the eigenvectors. In this way, rare events, if identified reliably, can be used to detect low dimensional deterministic behavior. Indeed, Ott, Grebogi, and Yorke (OGY) have developed a chaos control algorithm based upon PUP's [9], which has been demonstrated in a variety of physical systems [10]. Another recent approach has been proposed by Kaplan [11]. These new methods, wherein rare events or their signatures are detected, should be contrasted with traditional methods which rely on, for example, dimension estimates based on the averaged stationary properties of chaotic attractors and which subject the entire data file to analysis.

Recently, the OGY algorithm has been used in a spectacular set of experiments to detect chaos and to control it in a rabbit heart and in a hippocampal slice [12]. However, encounters with a PUP, upon which all of the work referenced in [10] and [12] is based, can also be detected in noisy limit cycles [13], and even in files made up from purely random distributions: Gaussian, Poisson, and gamma [14]. But the question is, *How often* can such encounters be detected? Therefore, the reliable detection of chaos by encounters with PUP's becomes a matter of statistics.

In this Letter we show that chaotic attractors can be detected reliably by using these methods applied to two example systems: a periodically forced Van der Pol oscillator and a bistable, first order time delay system, both with significant amounts of added noise. Both systems are realized as analog simulators and thus output continuous voltages. But data obtained from biological preparations most often are a series of interspike time intervals. To obtain such a series from the continuous outputs, we follow Sauer [15] and pass the voltages through a threshold detector which outputs a narrow pulse each time a positive-going threshold crossing occurs (negative-going return crossings are ignored). Thus the continuous outputs of the simulators mimic a neuron membrane potential: Whenever the threshold is exceeded, the neuron fires. Each record of interspike time intervals is first analyzed for encounters, then the sequential order of the intervals is randomly scrambled to form a surrogate set and reanalyzed. The number of detected encounters in the original set N is then compared to the number detected in the surrogate set N_s, with the fraction

$$F = \frac{N - N_s}{N + N_s}. \tag{1}$$

Scrambling was accomplished using either a Gaussian or a uniform distribution. Neither the choice of distribution nor the threshold level [16] (so long as the threshold lay well within the attractor) had any measurable effect on the results.

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Our experiment consists of measuring F as a function noise intensity for two different systems. Assuming that the surrogate data sets yield no encounters, then $F = 1$ cleanly indicated the presence of a PUP. Of course, the selection criteria by which a valid encounter is defined are all important, and we show below that, even with more stringent criteria, encounters are detected for both chaotic and noisy limit cycles and their surrogates.

A recording consisted of a set of time intervals between pulses from the threshold detector. Each record consisted of approximately 4×10^4 time intervals obtained in about 3.5 min of recording time. The records were then analyzed by plotting a first-return map, T_n vs T_{n-1}, searching the record for encounters as defined by one of three selection criteria, and then assembling and searching the surrogate record for encounters. The line of slope 1 (45° line) on the return map is the line upon which all period 1 orbits lie. The selection criteria were as follows. *Level 0*, three sequential points which approach the 45° line at successively decreasing perpendicular distances, followed by three points which depart at successively increasing distances. *Level 1*, the same but with straight lines fit to the three approaching points and to the three departing points with a slope (eigenvalue) condition applied, $0 > m_s \geq -1$ for the approaches along the stable manifold, and $-1 > m_{us} > -\infty$ for departures along the unstable manifold. *Level 2*, the same but with the additional condition that the intersection of the straight line fits must lie within a circle of specified radius centered on the 45° line at the location of the PUP, which was determined by the maximum of a histogram of all time intervals. These criteria define encounters with increasing rigor in order to demonstrate the statistical effects of such definitions.

An example taken from the Van der Pol system with small noise is shown in Fig. 1(a). The open circles are the return map. Three encounters, selected at level 1 and found at widely separated locations in the record, are shown by the solid triangles; the upright triangles show the stable manifold and the inverted triangles show the unstable manifold. The solid lines with negative slope are least square fits to the encounter data and indicate the stable and unstable directions. The three encounters are similar, that is, the solid triangles cluster in groups of three, and this is consistent with the observations of Schiff *et al.* [12(b)]. The map of the surrogate record is shown in Fig. 1(b). Three encounters satisfying the level 1 criteria are shown but are not of the same quality as those from the original record. They are recorded nevertheless in order to avoid subjective selection. Figure 1(c) shows the effect of the selection level on the Van der Pol data, where we have plotted F versus the intensity for level 0, diamonds; level 1, circles; and level 2, inverted triangles. We note that F is close to one (near perfect discrimination of the noisy chaotic attractor from its surrogate) for small noise, and that it remains distinguishable ($F > 0$) even for the largest noises measured, and that the distinguishability is improved in the case of the more stringent selection criteria. Moreover, even for the lowest level of selection, $F > 0$, at least over the range measured here. This is a remarkable result. It means that a chaotic attractor can be detected even in the presence of large noise by searching for certain events which, though rare, meet specific criteria. The implication of this result is that such attractors may be found reliably in records of length commonly obtained in biological experiments. We now turn to a description of the systems and the experiments performed with them.

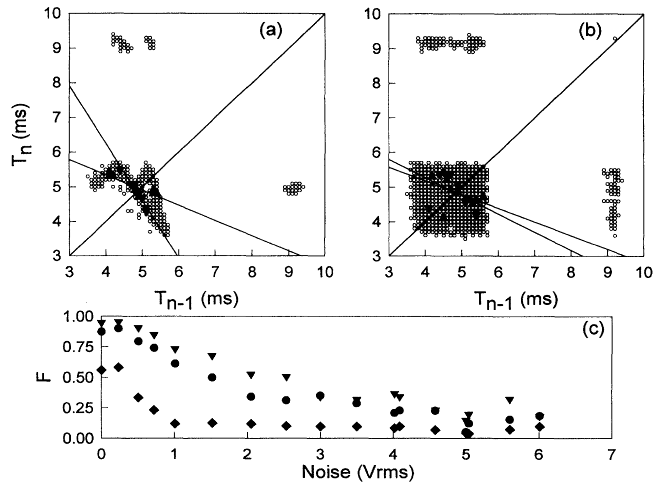

FIG. 1. (a) A return map (open circles) measured for the chaotic attractor of the Van der Pol oscillator with a noise voltage of 0.25 V rms, denoting a set of three encounters showing approaches along the stable manifold (upright solid triangles) and departures along the unstable manifold (inverted triangles). The negative slope lines are fits to the approach and departure data sets. Level 1 selection criteria were used (see text). (b) A set of three encounters taken from the surrogate of the record used for (a). (c) The fraction F plotted versus noise voltage for the chaotic Van der Pol attractor for selection level 0 (diamonds), 1 (circles), and 2 (inverted triangles). $F > 0$ implies positive detectability of the PUP.

The Van der Pol simulator was designed to mimic the equation

$$\tau_i^2 \ddot{u} - \tau_i(\epsilon - u^2)\dot{u} + u + A\cos\omega t + \xi(t) = 0, \quad (2)$$

where τ_i is the characteristic system time (the integrator time constant), ϵ is the nonlinearity parameter, $A\cos\omega t$ is the periodic forcing, and $\xi(t)$ is the noise [16]. For certain pairs of values A and ω, either a chaotic attractor or a nearby limit cycle can be obtained. The response of this system, $u(t)$, was passed through a threshold detector to make time interval records.

The object is to study the reliability of detection of the chaotic attractor in comparison with that of a noisy limit cycle. Figure 2 shows the data obtained for level 1 selection, where the solid circles represent the chaotic attractor and the solid triangles the limit cycle. The fraction F is plotted in Fig. 2(a) whereupon comparing circles to triangles we note that good discrimination is obtained over at least half the range, perhaps up to about 3 V rms of noise. For comparison, in the absence of noise, the chaotic response $u(t)$ was about 1.7 V rms. Thus the degree of noise contamination was about 1.76 times the magnitude of the uncontaminated response. Each open symbol is the mean of six independent records obtained at that noise voltage, and the error bars are the standard deviations. True, the very first triangle, for 0.25 V rms noise, shows a large F value comparable to that of the chaotic attractor. This is an artifact. The algorithm searches for signatures of PUP's. There can be none in the data file from a low noise limit cycle. Adding noise to the limit cycle increases the randomness, resulting in more detected chance encounters (for example, that triangle resulted from the detection of only one encounter in the record, and zero in its surrogate). Figures 2(b) and 2(c) show the actual number of encounters detected in the records and in their surrogates, respectively. The number of detected encounters for the limit cycle data (triangles) increases with increasing randomness in both data files [Fig. 2(b)] and their surrogates [Fig. 2(c)].

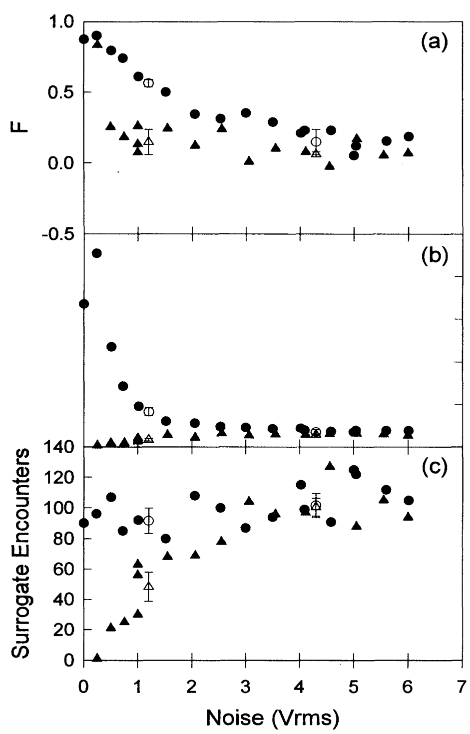

FIG. 2. Data for the Van der Pol oscillator versus noise voltage: chaotic attractor (solid circles); cycle (solid triangles) using selection level 1 (see text). (a) The fraction F. (b) The total number of detected encounters in each record of 4×10^4 time intervals. (c) The same, but for the surrogates of the records used in (b). The open symbols with error bars indicate the statistical scatter (see text). The root mean square noise voltages plotted on the abscissa are $\sqrt{\langle \xi^2 \rangle}$, where ξ is the noise input in Eq. (2).

The Van der Pol system is relatively robust to additive noise. In order to investigate a system which is more sensitive to the noise and, moreover, is high dimensional, we studied the time delay system,

$$\tau_i \dot{x}(t) = x(t - \tau_d) - x^3(t - \tau_d) + \xi(t), \qquad (3)$$

where τ_d is the time delay and the system's only bifurcation parameter. For $\xi = 0$, this system bifurcates to a local limit cycle confined to the bottoms of the potential wells at $\tau = 1.179$, displays a homoclinic orbit which just touches the central unstable point at $\tau = 2.110$, and then shows a global limit cycle until $\tau = 2.427$ which marks the bifurcations to chaos [17]. We operate this system at $\tau = 2.392$ for the limit cycle and at $\tau = 2.472$ for the chaotic attractor. Note the very small difference between these values. Because these orbits lie close to the homoclinic orbit, this system is very sensitive to added noise. In this case we can achieve very substantial changes in the attractor by adding only 100 mV rms of noise. The data are shown in Fig. 3(a) where the solid circles represent the noisy chaotic attractor and the triangles the limit cycle. As before, the open symbols are the means of six independent records. We note that F is now negative for

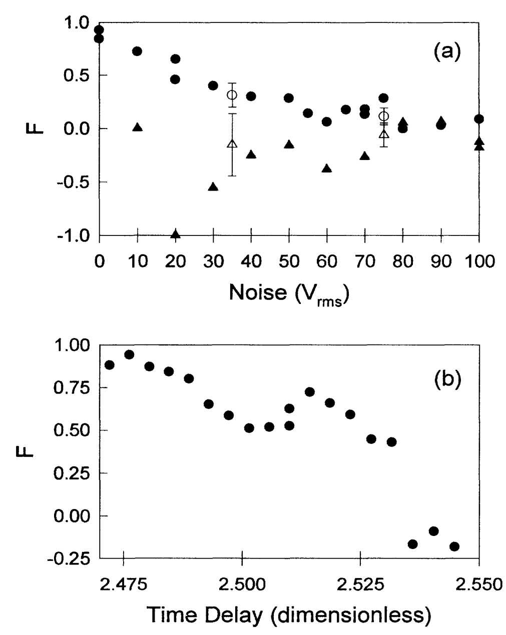

FIG. 3. Data for the time delay system for the chaotic attractor (solid circles) and limit cycle (solid triangles) using selection level 1. (a) The fraction F versus the rms noise voltage $\sqrt{\langle \xi^2 \rangle}$, where ξ is the noise input in Eq. (3). (b) The fraction F for the chaotic attractor of the noise free system versus time delay. Increasing time delay represents increasing dimension, which plays the same general rule for the reliability of detection of PUP's as additive noise.

the limit cycle. This means that there were *fewer* selected encounters in the record than in its surrogate, but the number of encounters in the surrogate grows with increasing noise so that $F \rightarrow 0$ again for large noise. *This result clearly demonstrates the ability of this method to distinguish between noisy limit cycle and chaotic attractors.* We note that statistically good discrimination is possible up to about 70 mV rms of noise.

High dimensionality is thought to defeat efforts to detect chaos in biological preparations. Our time delay system is potentially infinite dimensional, but the actual fractal dimension of the noise free system depends on the time delay [18]. We have tested our level 1 selection criteria as a function of time delay in the chaotic regime of the noise free ($\xi = 0$) system. The results are shown in Fig. 3(b), where it is evident that increasing time delay has the same overall effect as increasing noise, that is, it degrades the quality of discrimination. Moreover, we see that only *very small* increases in time delay are sufficient to bring about zero or negative values of F [19]. Thus we find our selection algorithms of little use for high dimensional systems (but how high we cannot say).

We conclude that chaos can be detected in records of noisy attractors by the appropriate selection of rare events, and that it can be reliably distinguished from noisy limit cycles.

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[16] The threshold level was set to $V_{\text{th}} = 1.0$ V compared to the magnitude of the noise free peak response of $V_{\text{pk}} = \pm 2.5$ V for the limit cycle, and a range of ± 1.5–3.2 V for the chaotic attractor. Thus, without added noise, a single threshold crossing was guaranteed for every circuit around the attractor. The parameters of the Van der Pol oscillator were $\tau_i = 8.3 \times 10^{-4}$ s, $\epsilon = 1.5$, $A = 5.1$ V pk $f(= w/2\pi) = 209.4$ Hz, and $\sqrt{\langle \xi^2 \rangle}$ varied from 0 to 6 V rms. For $A = \xi = 0$, the limit cycle oscillation was 1.7 V rms at 172.5 Hz. With $A \neq 0$, a simple chaotic attractor was observed. The noise is quasiwhite, meaning that its correlation time $\tau_n(= 3.2 \times 10^{-4}$ s$) < \tau_i$.

[17] The time delay was achieved with a "bucket brigade" device (EG&G type RD5106A). For easier comparison with numerical simulations and theory, we use the dimensionless time delay $\tau = \tau_d / \tau_i$. The noise had the same bandwidth as for the Van der Pol experiment, but its amplitude now is in the millivolt range.

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[19] The noise free time delay system is quite complex. The structure of F, as shown in Fig. 3(b), is reproducible. Moreover, for a larger range of τ values, various structures appear including windows where $F \rightarrow 1$.