Renormalization Group Theory of Hysteresis

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We apply renormalization group theory directly to the first-order phase transition of the large-N model driven linearly by an external magnetic field $H = \dot{H}t$, where $\dot{H}$ is the sweeping rate. Novel dynamic scaling forms for the magnetization, the structure factor, and the area of hysteresis loop are $M(\dot{H}, t, T) = f(\dot{H}^{1/2}t, \dot{H}^{(d-2)/4}T)$, $C(k, \dot{H}, t, T) = \dot{H}^{-d/4}f'(\dot{H}^{-1/4}k, \dot{H}^{1/2}t, \dot{H}^{(d-2)/4}T)$, and $A = \dot{H}^{1/2}g(\dot{H}^{(d-2)/4}T)$, respectively, where T is the temperature of the system, d the spatial dimensionality, k the wave number, and f, f', and g scaling functions. These results show that the rate of the external driving field can serve as a scaling parameter to study hysteresis.

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Hysteresis is a ubiquitous phenomenon in first-order phase transitions. It is, however, a complex process of a dynamic and nonlinear nature that eludes serious treatment, both experimentally and theoretically. There is an empirical law by Steinmetz [1], dating back to last century, showing that on real magnetic systems the area of typical hysteresis loops A is given by

$$A \approx H_0^{1.6}, \tag{1}$$

where H_0 is the amplitude of the oscillating external magnetic field. There are also a geometrical theory of hysteresis [2], a statistical theory of the nonlinear relaxation function to describe metastable decay [3], and a hysteresis criterion based on rate competition [4]. Only recently, however, is there an increasing interest [5–15] in it since the rediscovery of the scaling law, Eq. (1), in the large-N model by Rao, Krishnamurthy, and Pandit [5].

Most current work on the scaling of hysteresis has concentrated on the energy dissipation in a cycle, represented by the area of the loops. Theoretical [5,6,8–10,12–15] and experimental [7,11] results have demonstrated that the energy dissipation per cycle varies as a power law as

$$A \approx H_0^{\alpha}\Omega^{B} \tag{2}$$

for low amplitude H_0 and frequency Ω of the sinusoidal driving field. In this limit of low H_0 and Ω, the field is a linear function of time t with a proportionality coefficient $H_0\Omega$ and thus $\alpha = \beta$. Therefore, for the directly linear driving field with a sweep rate $\dot{H}$ [12,15],

$$H = \dot{H}t, \tag{3}$$

$$A \approx \dot{H}^{\alpha}. \tag{4}$$

Logarithmic corrections to the scaling [8,12] and dynamic scaling have also been proposed [13].

For the two-dimensional (2D) Ising model [6] and a cellular dynamic system [9], it was found that $\alpha \sim 0.46$ and $\beta \sim 0.36$. For mean-field models [7,15] and the double hysteresis loops in the Φ^6 model in the large-N limit [15], $\alpha = \beta = \frac{2}{3}$, with a nonzero adiabatic area [14,15]. In most regions of the phase diagram of the large-N model, which is of concern here, qualitative analysis [8] and singular perturbation theory [12] all produce $\alpha = \beta = \frac{1}{2}$, and this is also confirmed by direct numerical integration [8,12,15].

As phase ordering (in zero external field) of the same system is governed by the zero-temperature fixed point [16], it seems plausible that the driven first-order phase transition between the "up" and "down" magnetizations is also controlled by the same fixed point. In fact, it has been shown [12] that the whole dynamics, not just the exponents, is universal, independent of the particular form of the free energy. Also, for a system of 2D planar spins, the exponent α, like the corresponding critical exponent in the same system, has been shown to vary continuously with temperature [10].

The universality of the dynamics and exponents suggests that a renormalization group (RG) theory might be fruitful. Because of the continuous symmetry, there are gapless Goldstone modes. As pointed out in Ref. [12], as $\dot{H} \to 0$, the characteristic length of the system diverges, and fluctuations affecting the magnetization $M(t)$ inside each domain partially cancel. This is the underlying picture of the renormalization analysis. Noticing that RG theory has been successfully applied to spinodal decomposition and phase ordering [16,17] but not directly to hysteresis, following the same line of reasoning, we show for the first time in this paper that RG may apply as well to driven transitions, which have been proved to be a good context to study hysteresis. The principal new ingredient is that, for a system linearly driven by an external field given in Eq. (3), $\dot{H}$ has also to be scaled as $\dot{H}' = b^{2z-y}\dot{H}$, where $-y = 0$ and $-z = -2$ are the eigenvalues of time and real space order parameter at the zero-temperature fixed point, respectively. As a consequence, the magnetization and the structure factor can be cast into nicely scale invariant forms, namely,

$$M(\dot{H}, t, T) = f(\dot{H}^{1/2}t, \dot{H}^{(d-2)/4}T), \tag{5}$$

$$C(k,\dot{H},t,T) = \dot{H}^{-d/4} f'(\dot{H}^{-1/4}k, \dot{H}^{1/2}t, \dot{H}^{(d-2)/4}T), \tag{6}$$

where T is the temperature of the system, d the spatial dimensionality, k the wave number, and f and f' scaling functions. Numerical results in Fig. 1 show that the original curves in (a) for different $\dot{H}$ and T collapse completely into one single line in the transition region (owing to the difficulty in clarity in the presentation of the overlap of 3D surfaces, the structure factor is not shown here). Note that the curve with $\dot{H} = 0.005$ is *not* expected to overlap with the others, because its temperature has not been rescaled. For sufficiently low temperature and $\dot{H}$, the area of the hysteresis loops $\int M\, dH$ per cycle can be readily obtained from Eq. (5) to be

$$A = \dot{H}^{1/2} g(\dot{H}^{(d-2)/4}T), \tag{7}$$

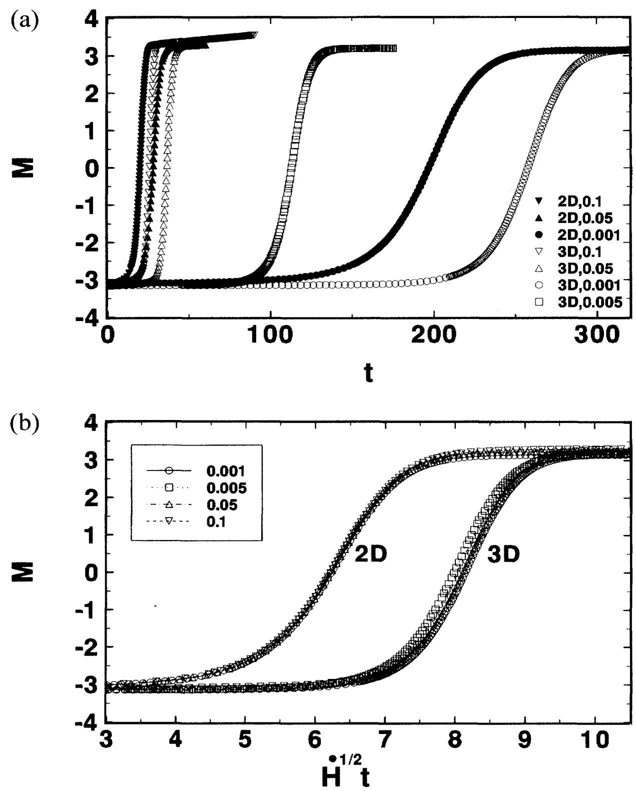

FIG. 1. (a) Curves of magnetization vs time, for different sweeping rates in 2 and 3 dimensions as indicated in the legend. For the 2D curves, all temperatures T are set to 1; for the 3D ones with $\dot{H} = 0.1$ and 0.05, however, $T = (0.0001/\dot{H})^{1/4}$, i.e., relative to that of $\dot{H} = 0.001$, $T = 1$. The 3D curve with $\dot{H} = 0.005$, on the other hand, has the same $T = 1$. (b) Scaled curves of those given in (a) in 2D and 3D. Note that the curve with $\dot{H} = 0.005$ has the same temperature as that with $\dot{H} = 0.001$ and is *not* expected to collapse to the latter. It is given only for comparison and to show the effect of temperature.

where the scaling function $g(x)$ is a small correction to the scaling in Eq. (4). Moreover, these results are independent of the detailed form of the free energy.

We start from the Langevin equation for the N-component order parameter $\Phi(\mathbf{x})$,

$$\frac{\partial \Phi(\mathbf{x},t)}{\partial t} = -\lambda \frac{\partial F[\Phi]}{\delta \Phi(\mathbf{x},t)} + \eta(\mathbf{x},t), \tag{8}$$

where $\eta(\mathbf{x},t)$ is a Gaussian white noise with zero mean and correlator

$$\langle \eta_i(\mathbf{x},t)\eta_j(\mathbf{x}',t)\rangle = 2\lambda T \delta_{ij}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{9}$$

and the Ginzburg-Landau-Wilson free energy functional is given by

$$F[\Phi] = \frac{1}{2}\int d^dx \Big[c(\nabla\Phi)^2 + r\Phi^2 + \frac{u}{2N}(\Phi^2)^2 - 2\sqrt{N}\,\mathbf{H}\cdot\mathbf{\Phi}\Big], \tag{10}$$

where r, c, and u are coupling constants and λ is the kinetic coefficient. In the large-N limit, the dynamic equation reduces to [17,18]

$$\frac{dM}{dt} = -\lambda(RM + H), \tag{11a}$$

$$\frac{\partial C_\perp(k,t)}{\partial t} = 2\lambda T - 2\lambda(ck^2 + R)C_\perp(k,t), \tag{11b}$$

with

$$R = -(r + uS + uM^2), \tag{11c}$$

$$S = \int \frac{d^dk}{(2\pi)^d} C_\perp(k,t), \tag{11d}$$

where $C_\perp(k,t)$ is the transverse structure factor (the $\perp$ will be dropped below) and the integral is cut off at Λ.

Consider a system which initially equilibrates in a negative external field. It is then driven by the field given in Eq. (3) with a small positive $\dot{H}$, such that, when $t = 0$, $H = 0$. Following the standard procedures of RG, in particular Ref. [17], first the "hard" modes with $\Lambda/b < k < \Lambda$ $(b > 1)$ are eliminated by solving the Langevin equation for the time evolution of these modes, and substituting the solution into the equation for the "soft" modes with $k < \Lambda/b$. The only effect after this stage is that, for the dynamic equation of the soft modes, R changes to

$$R(t) = r + uM(t)^2 + uS_s(t) + uS_h(t), \tag{12}$$

where the subscripts s and h refer to integration over the soft and hard modes, respectively. For small sweeping rates, the structure factor of the hard modes can follow its steady variation even beyond $H = 0$. As a consequence,

$$C_h(k,t) \approx \frac{T}{ck^2 + R_{\text{steady}}} \approx \frac{T}{ck^2 + H/M}. \tag{13}$$

Therefore, for small rates and hence small H, H/M can be neglected compared to the large k of the hard modes and we have

$$S_h(t) \sim TK_c(1 - b^{2-d})/c\,, \tag{14}$$

where $K_c = 2^{-d+1}\pi^{-d/2}\Lambda^{d-2}/(d-2)\Gamma(d/2)$ and Γ is the gamma function. Thus no new terms are generated by the coarse graining.

Next the soft modes are rescaled via $k = k'/b$ in order to restore the ultraviolet cutoff for these modes to its original value Λ, while time t is simultaneously rescaled via $t = b^z t'$ as required by scale invariance. We adopt the convention of [19], i.e.,

$$k = bk', \qquad x = x'/b, \qquad t = b^z t', \tag{15a}$$

$$\Phi(\mathbf{x}) = b^y \Phi'(\mathbf{x}'), \qquad \Phi_k = b^{y+d/2}\Phi'_{k'}\,. \tag{15b}$$

Accordingly,

$$\begin{aligned} M(t) &= \langle \Phi \rangle = b^y M'(t')\,, \\ C(k,t) &= \langle \Phi_k(t)\Phi_{-k'}(t)\rangle = b^{d+2y} C'(k',t')\,, \\ S(t) &= b^{2y} S'(t')\,. \end{aligned} \tag{15c}$$

Substituting Eqs. (15) into Eqs. (11), taking into account Eq. (3) and rewriting the resultant equations in the original form with primed parameters replacing the original ones, we obtain as the recursion relations

$$c' = b^{z-2}c\,, \tag{16a}$$

$$T' = b^{z-d-2y}T\,, \tag{16b}$$

$$\dot{H}' = b^{2z-y}\dot{H}\,, \tag{16c}$$

$$R'(t') = r' + u'M'^2(t') + u'S'(t') = b^z R(b^z t)\,, \tag{16d}$$

and with the help of Eqs. (12) and (14) we have two further relations

$$u' = b^{z+2y}u\,, \tag{16e}$$

$$\tau' = b^z \tau\,, \tag{16f}$$

where $\tau = r + uTK_c/c = r(T_c - T)/T_c$ and $T_c = -rc/uK_c$.

For the zero-temperature fixed point, it is required that c and τ/u be finite [17], which leads to

$$z = 2, \qquad y = 0\,. \tag{17}$$

Consequently, $\tau' = b^2\tau$ and $u' = b^2 u$. Coefficients of higher order terms in the Hamiltonian, such as $(\Phi^2)^3$, also transform in the same way, besides renormalizing the lower ones, and so the Hamiltonian scales as b^{d-2}. Therefore, the results are independent of the detailed form of $F[\Phi]$, which has been confirmed [12,15].

As the transformation of the Hamiltonian can be taken into account by the inverse transformation of temperature b^{2-d} [16], we need consider only temperature and, therefore, we have

$$\begin{aligned} M(\dot{H},t,T) &= M'(\dot{H}',t',T') = M(\dot{H}',t',T') \\ &= M(b^4\dot{H}, b^2 t, b^{2-d}T)\,, \end{aligned} \tag{18a}$$

$$\begin{aligned} C(k,\dot{H},t,T) &= C'(k',\dot{H}',t',T') = C(k',\dot{H}',t',T') \\ &= b^d C(k/b, b^4\dot{H}, b^2 t, b^{2-d}T)\,. \end{aligned} \tag{18b}$$

Setting $b = \dot{H}^{-1/4}$ (>1), we finally obtain the central results of this paper, Eqs. (5) and (6).

It is readily seen that for $d = 2$ T is invariant and accordingly α is exactly equal to $\frac{1}{2}$. For $d > 2$, when T and $\dot{H}$ are sufficiently low, $\dot{H}^{(d-2)/4}T$ is small, and α also approaches $\frac{1}{2}$. For larger $\dot{H}$, T has to be concurrently lowered to reduce thermal fluctuation in order to obtain morphology similar to the smaller $\dot{H}$ ones. These results are completely confirmed on referring to Fig. 1. One point should be made, however, and this is that in the limiting case $T = 0$ $\dot{H}$ must diverge and accordingly the transition field shifts to a finite value, the spinodal point (H_s, M_s), and the dynamics reduce to the mean-field result, because the equilibrium structure factor is equal to zero identically. In this case, expanding the magnetization at the spinodal point as in Ref. [7], and then applying the rescaling procedures described above, we arrive at a scale invariant form for the magnetization such as

$$M(\dot{H},t) \sim M_s + \dot{H}^{1/3} f(\dot{H}^{1/3}t)\,, \tag{19}$$

where f is also a scaling function. Thus α crosses over to $\frac{2}{3}$ [7].

The relevance of the theoretical hysteresis loops to the experiments and previous theories has been discussed at length in Ref. [5]. We would just point out that the highly simplified large-N model cannot be expected to agree quantitatively with experiment. Deliberate experiments, however, have appeared that come closer to the theories. Therefore such attractive scaling forms as Eqs. (5)–(7) will quite probably stimulate further experimental and theoretical investigations that finally will get hysteresis under control.

In conclusion, we would like to emphasize the idea that can be extracted from this work. Hysteresis is essentially an irreversible process. Therefore, a complete description should take into account the variable of the process. The RG approach and its resultant dynamic scaling forms show that the rate of the linear driving field can serve as a variable not only for characterizing but also for scaling the properties of hysteresis. The underlying physical picture is that the morphology of different scanning rate is similar. This might open a new way to approach hysteresis.

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