Universal Scaling Functions in Critical Phenomena

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(Received 22 February 1995)

A histogram Monte Carlo method is used to evaluate the existence probability E_p and the percolation probability P of bond and site percolation on finite square, plane triangular, and honeycomb lattices. We find that, by choosing a very small number of nonuniversal metric factors, all scaled data of E_p and P may fall on the same universal scaling functions. We also find that free and periodic boundary conditions share the same nonuniversal metric factors. This study may be extended to many critical systems.

PACS numbers: 05.50.+q, 75.10.–b

Scientific researches can be considered as processes of data reduction, in the sense that one looks for new ideas and principles such that a variety of data can be represented by one or a few equations, or functions, with a small number of parameters. For example, in the theory of critical phenomena, the idea of universality leads to a set of fixed critical exponents in many different systems [1,2]. The idea of the universal finite-size-scaling functions proposed by Privman and Fisher [3] represents another important and interesting example. In this paper we will present results which support, and generalize, the idea of Privman and Fisher on universal scaling functions.

Finite-size scaling is important in both theoretical [4–6] and experimental [7] studies of critical phenomena. According to the theory of finite-size scaling [4–6], if the dependence of a physical quantity Q of a thermodynamic system on a parameter t, which vanishes at the critical point $t = 0$, is of the form $Q(t) \sim t^a$ near the critical point, then for a finite system of linear dimension L, the corresponding quantity $Q(L,t)$ is of the form

$$Q(L,t) \sim L^{-ay_t} F(tL^{y_t}), \tag{1}$$

where y_t $(=\nu^{-1})$ is the thermal scaling power and $F(x)$ $(x = tL^{y_t})$ is the scaling function. It follows from (1) that the scaled data $Q(L,t)L^{ay_t}$ for different values of L and t are described by a single function $F(x)$. Thus it is important to know general features of the scaling function under various conditions.

In 1984, in a paper on finite-size scaling, Privman and Fisher [3] proposed the concept of universal scaling functions and nonuniversal metric factors. Specifically, they proposed that, near $t = 0$, the singular part of a free energy can be written as

$$f_s(t,L) \sim L^{-d} Y(DtL^{y_t}), \tag{2}$$

where d is the spatial dimensionality of the lattice, Y is a universal scaling function, and D is a nonuniversal metric factor [3,8]. Following this idea, Lee [8] has recently evaluated the scaling function and the nonuniversal metric factor for the three-state Potts model on the square lattice. According to the idea of universality [1,2], different systems in the same spatial dimensionality and having the same Hamiltonian symmetry share the same set of critical exponents. However, it seems that there have been no published results which show that many different systems in the same universality class [1,2] share the same set of universal scaling functions [3].

In this Letter, we use a histogram Monte Carlo simulation method (HMCSM) [9–14] to evaluate the existence probability $E_p(G,p)$ and the percolation probability $P(G,p)$ of bond and site percolation on the square (sq), the planar triangular (pt), and the honeycomb (hc) lattices. Here $E_p(G,p)$ is the probability that the system percolates. In the limit of $L \to \infty$, $E_p(G,p)$ approaches the step function $\theta(p - p_c)$ [6], where p_c is the critical probability. $P(G,p)$ is the fraction of lattice sites in the largest cluster in G, which is percolating; it is the order parameter of the system. $E_p(G,p)$ and $P(G,p)$ may be used in a percolation renormalization group method to calculate the critical point, critical exponents, and the thermodynamic order parameter for the percolation problem [13]. More precise definitions of $E_p(G,p)$ and $P(G,p)$ used in this Letter will be given below. It should be noted that $E_p(G,p)$ cannot be derived from the *free energy* of the system. Therefore, we study a problem which extends the scope considered by Privman and Fisher [3]. We find that by choosing an appropriate aspect ratio, i.e., width-to-high ratio, for each lattice and a very small number of nonuniversal metric factors for each model, the scaled data of E_p and P of all models with the same boundary conditions fall on the same curves. We also find that free and periodic boundary con-

0031-9007/95/75(2)/193(4)$06.00

ditions share the same nonuniversal metric factors. Thus, our results support, and generalize, Privman and Fisher's idea of universal scaling functions.

Here we briefly review the HMCSM for the site percolation [11–13] and define related quantities. The extension to the bond percolation [9,14] is straightforward. Our HMCSM is different from the one used by Gould and Tobochnik [15], in which only $E_p(G,p)$ is calculated. In the site percolation on a d-dimensional lattice G of N sites, each site of G is occupied with a probability p, where $0 \leq p \leq 1$. A cluster which extends from a given side of G to the opposite side is a percolating cluster. The subgraph whose largest cluster is percolating is a percolating subgraph and denoted by G_p', otherwise the subgraph is a nonpercolating subgraph. Then we have the definitions

$$E_p(G,p) = \sum_{G_p' \subseteq G} p^{v(G_p')}(1-p)^{N-v(G_p')}, \quad (3)$$

$$P(G,p) = \sum_{G_p' \subseteq G} p^{v(G_p')}(1-p)^{N-v(G_p')} N^*(G_p')/N, \quad (4)$$

where $v(G_p')$ is the number of occupied sites in G_p'. The summations in (3) and (4) are over all subgraphs G_p' of G, and $N^*(G_p')$ is the total number of sites in the largest cluster of G_p' [11–14]. We choose w different values of p. For a given $p = p_j$, $1 \leq j \leq w$, we generate N_R different subgraphs G'. The data obtained from the wN_R different G' are then used to construct three arrays of numbers of length N with elements $N_p(v)$, $N_f(v)$, and $N_{pp}(v)$, $0 \leq v \leq N$, which are, respectively, the total numbers of percolating subgraphs with v occupied sites, nonpercolating subgraphs with v occupied sites, and the sum of $N^*(G_p')$ over subgraphs with v occupied sites. In the large number of simulations, the existence probability E_p and the percolation probability P at any value of the site occupation probability p can then be calculated approximately from the following equations [9,11]:

$$E_p(G,p) = \sum_{v=0}^{N} p^v (1-p)^{N-v} C_v^N \frac{N_p(v)}{N_p(v)+N_f(v)}, \quad (5)$$

$$P(G,p) = \frac{1}{N} \sum_{v=0}^{N} p^v (1-p)^{N-v} C_v^N \frac{N_{pp}(v)}{N_p(v)+N_f(v)}, \quad (6)$$

where $C_v^N = N!/(N-v)!\,v!$.

We first use (5) and (6) and similar equations for bond percolations to evaluate the existence probability $E_p(G,p)$ and the percolation probability $P(G,p)$ for site and bond percolation on the pt, sq, and hc lattices with free boundary conditions. In such boundary conditions and in the limit $L \to \infty$, it has been found that for site and bond percolation on the sq lattice $E_p(G,p_c) = 0.5$ [16,17], and it has been proposed [18] that for bond and site percolation on the pt lattice with aspect ratio $\sqrt{3}/2$ and on the hc lattice with aspect ratio $\sqrt{3}$, $E_p(G,p_c)$ is equal to that for the square lattice, i.e., 0.5. Therefore, we choose a 433×500 pt lattice whose aspect ratio 433/500 is very close to $\sqrt{3}/2$, and a 433×250 hc lattice whose aspect ratio 433/250 is very close to $\sqrt{3}$. For the pt and hc lattices, L is given by $\sqrt{N}$. For the square lattice, the linear dimension L is chosen to be 512. The calculated results of E_p and P are shown in Figs. 1(a) and 1(b) by solid and dotted lines for site and bond percolation, respectively. Hu [11] has found that different boundary conditions give quite different scaling functions near the critical region. However, they give the consistent critical point, critical exponents, and the thermodynamic order parameter from renormalization group calculations [11]. Therefore, we also use the HMCSM to calculate the $E_p(G,p)$ and $P(G,p)$ for site and bond percolations

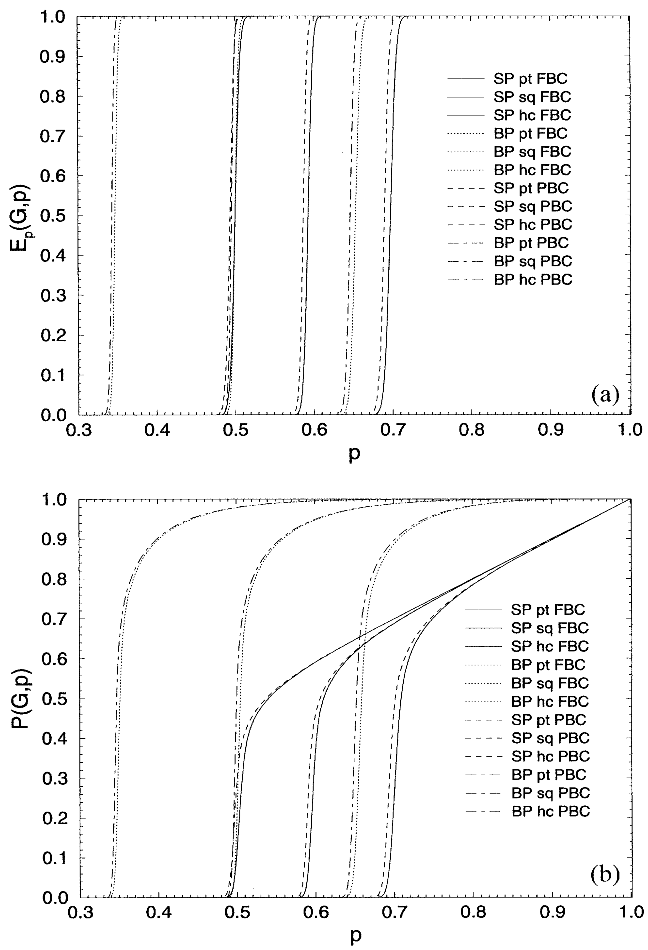

FIG. 1. Results for site percolation (SP) and bond percolation (BP) on the plane triangular (pt), square (sq), and honeycomb (hc) lattices. The solid (dotted) lines from left to right are for site (bond) percolations on pt, sq, and hc lattices with free boundary conditions (FBC). The dashed (dot-dashed) lines from left to right are for site (bond) percolations on pt, sq, and hc lattices with periodic boundary conditions (PBC). (a) E_p as a function of p. (b) P as a function of p.

on a 512×512 sq lattice, a 433×500 pt lattice, and a 433×250 hc lattice, and such lattices have periodic boundary conditions. The calculated results are shown in Figs. 1(a) and 1(b) by dashed and dot-dashed lines for site and bond percolation, respectively.

For bond and site percolation on planar lattices, it is generally believed that the *exact* y_t and the order parameter exponent β are 3/4 and 5/36, respectively [6]. It is also believed that the exact critical point p_c for the bond percolation on the sq, pt, and hc lattices are, respectively, 0.5, 0.347 296..., and 0.652 703... [6], and the exact p_c for the site percolation on the pt lattice is $\frac{1}{2}$ [6]. Ziff and Langlands *et al.* have carried out extensive Monte Carlo simulations to obtain $p_c = 0.5927460 \pm 0.0000005$ [17] and $p_c = 0.697034 \pm 0.000006$ [18] for the site percolation on the sq and hc lattices, respectively. Using the aforementioned exact values of y_t, β, and p_c [6], and numerical values of p_c [17,18], we first obtain $E_p(G,p)$ and $P(G,p)/L^{-\beta y_t}$ as a function of $z = (p - p_c)L^{y_t}$. We then use application programs XVGR in Sun workstations to fit such data as polynomials of z. The coefficients of the linear terms for $E_p(G,p)$ are used to calculate D_1, and the coefficients of the constant and linear terms for $P(G,p)/L^{-\beta y_t}$ are used to calculate D_3 and D_2, respectively. The calculated values of D_1, D_2, and D_3 are shown in Table I, where the notations for periodic boundary conditions are represented by D_1', D_2', and D_3' [19]. We have plotted the data for $E_p(G,p)$ of Fig. 1(a) as a function of $x = D_1(p - p_c)L^{y_t}$ in Fig. 2(a), and $D_3P(G,p)/L^{-\beta y_t}$ for $P(G,p)$ of Fig. 1(b) as a function of $x = D_2(p - p_c)L^{y_t}$ in Fig. 2(b). Since the critical exponent of E_p is zero [6], there is no need to divide E_p by the factor $L^{-a y_t}$ to obtain the scaling function for E_p, now denoted by $F(x)$. It is obvious that $F(0) = E_p(G, p_c)$. The scaling function for $P(G,p)$ is denoted by $S(x)$.

Figures 2(a) and 2(b) show that E_p and P possess well-defined *universal scaling functions*. It is of interest to note that for each lattice D_1 is consistent with D_2 within numerical uncertainty and the values of D_1, D_2, and D_3 for the free boundary condition of a lattice are consistent with those for the periodic boundary condition of the same lattice within numerical errors. In other words, only a small number of nonuniversal metric factors are needed to reach the universal scaling functions shown in Figs. 2(a) and 2(b). For the free boundary condition, we find that $F(x) = 0.49(9) + 0.9(7)x - 1(0)x^3 + 1(0)x^5 + \cdots$, $S(x) = 0.39(2) + 1.0(3)x + 0.5(5)x^2 - 0.7(3)x^3 - 0.5(5)x^4 + \cdots$. For the periodic boundary condition, we find that $F(x) = 0.93(4) + 0.3(9)x - 0.9(1)x^2 + 0(7)x^3 + 0(7)x^4 - 1(5)x^5 + \cdots$, $S(x) = 0.94(7) + 0.9(9)x - 0.8(5)x^2 - 0.2(5)x^3 + 1(3)x^4 + \cdots$.

We have also studied site and bond percolations on a 256×512 rectangular lattice, a 216×250 hc lattice, and a 216×500 pt lattice; the aspect ratios of such lattices are about half of the lattices mentioned above. We have found that D_1, D_2, and D_3 for each of these new lattices are consistent with those listed in Table I of the corresponding lattice [20]. If the other factor is used to reduce the aspect ratios, similar results could be expected.

At present, the conformal theory is only applied to the bond percolation on rectangular lattices with free boundary conditions and only at the critical point p_c [16]. It is of interest to extend such studies to other lattices and for p away from p_c in order to calculate *exact* D_1 and then compare such values with our numerical values.

We expect that the features of universal scaling functions and nonuniversal metric factors found in this Letter may be applied to a variety of critical systems, e.g., classical and quantum spin models, lattice gauge models, spin glass, etc., where finite-size scalings may be applied [5]. In particular, it has been found that phase transitions of many Ising-type spin models and hard-core particle models are percolation transitions of the corresponding correlated percolation models [21–23]. We may extend the method of this Letter to calculate universal scaling functions for $E_p(G,p)$ and $P(G,p)$ of such models [24]. With

TABLE I. Nonuniversal metric factors for site and bond percolation on square (sq), plane triangular (pt), and honeycomb (hc) lattices. The values of w and N_R used in the simulations are also shown. w, N_R, D_1, D_2, and D_3 are for lattices with free boundary conditions; w', N_R', D_1', D_2', and D_3' are for lattices with periodic boundary conditions.

Model	Site	Site	Site	Bond	Bond	Bond
Lattice	sq	pt	hc	sq	pt	hc
w	420	290	310	318	318	318
N_R	56 000	45 000	40 000	50 000	20 000	30 000
D_1	0.786 ± 0.015	0.793 ± 0.025	0.858 ± 0.020	1	1.227 ± 0.044	0.944 ± 0.016
D_2	0.794 ± 0.017	0.792 ± 0.018	0.870 ± 0.020	1	1.240 ± 0.043	0.954 ± 0.018
D_3	1.485 ± 0.016	1.706 ± 0.017	1.292 ± 0.015	1	1.021 ± 0.021	0.987 ± 0.011
w'	420	290	310	318	318	290
N_R'	90 000	30 000	30 000	35 000	20 000	15 000
D_1'	0.785 ± 0.010	0.789 ± 0.011	0.870 ± 0.012	1	1.239 ± 0.021	0.974 ± 0.013
D_2'	0.792 ± 0.012	0.792 ± 0.012	0.879 ± 0.015	1	1.247 ± 0.027	0.979 ± 0.016
D_3'	1.476 ± 0.006	1.712 ± 0.007	1.289 ± 0.004	1	1.016 ± 0.007	0.987 ± 0.005

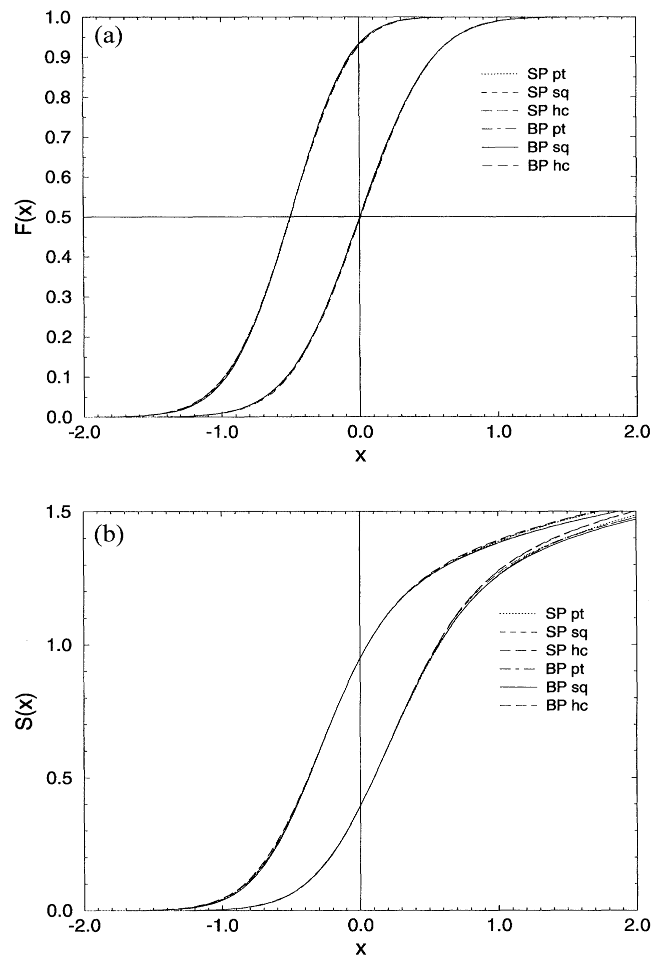

FIG. 2. (a) The calculated E_p for the site and bond percolation on pt, sq, and hc lattices as a function of x, where $x = D_1(p - p_c)L^{y_t}$. The scaling function is $F(x)$. The lower (upper) curves are for free (periodic) boundary conditions. (b) The calculated $D_3P/L^{-\beta y_t}$ for the site and bond percolations on pt, sq, and hc lattices as a function of x, where $x = D_2(p - p_c)L^{y_t}$. The scaling function is $S(x)$. The lower (upper) curves are for free (periodic) boundary conditions.

the rapid progress of computing and experimental facilities, more and more results of critical systems may be obtained and analyzed by finite-size scalings. The results of this Letter will greatly reduce the amount of jobs needed to obtain experimental or numerical data.

We are indebted to B. I. Halperin, K.-t. Leung, and Y. Saint-Aubin for useful discussions, and to F. Y. Wu for a critical reading of the manuscript. This work was supported by the National Science Council of the Republic of China (Taiwan) under Grant No. NSC 84-2112-M-001-013 Y, No. 84-0501-I-001-037-B12, and No. NSC 84-2112-M-001-048. The authors thank the Computing Center of Academia Sinica (Taipei), the National Center of High-speed Computing in Taiwan, and the Department of Physics of Harvard University for providing the computing and research facilities.

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