# Universal Properties of Multimode Laser Power Spectra

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(Received 12 April 1995)

In a multimode laser operating near steady state, we determine analytically relations which connect the power spectrum density of each modal intensity and of the total intensity at the same frequency. We prove that, if the laser is in an antiphase regime, these relations become independent of the initial condition. This property rests on the existence of widely different time scales for the oscillation frequencies and their damping. Numerical simulations indicate that these relations remain true when a small amplitude modulation is applied to the control parameter.

PACS numbers: 42.50.Ne, 42.55.Rz

Recently, multimode lasers have been intensively studied as examples of spontaneous self-organized time-periodic systems. This regime has been called antiphase dynamics (AD) in laser physics. It is a manifestation of the coherence property of nonsteady modal intensities that can be displayed by multimode lasers. It should not be confused with the electric field coherence of the single mode laser. AD has been reported in lasers in the case of spontaneous self-pulsing [1–3], in the presence of an external modulation [4,5], in the noise spectrum at steady state [6], in the transient relaxation to steady state [7,8], in the chaotic regime [9,10], and in the routes to chaos [11].

A laser oscillating on $N$ modes is characterized by $N$ modal intensities $I_n(t)$, $n = 1, 2, \ldots, N$. The rate equation limit, where only model intensities and population inversion are coupled, applies to all the lasers in which AD has been reported up to now. For such lasers, the sum of the modal intensities $\Sigma I(t) = \sum_{n=1}^{N} I_n(t)$ is the total intensity. In the case of self-pulsing, AD means that each modal intensity is periodic, though with different phases and/or frequencies, but the total intensity is also periodic. When the dynamics is characterized by the relaxation frequencies, AD means that each modal intensity is driven by a number of frequencies (smaller than or equal to the mode number) while the total intensity is driven by only one frequency, the one which is related to the single mode frequency. The purpose of this Letter is to put forward yet another signature of AD by deriving universal relations between the power spectra of $I_n(t)$ and $\Sigma I(t)$. Universality in this context means that the relations are independent of the initial condition, i.e., of the preparation of the system. We shall first show that, under rather weakly constraining conditions a general relation can be found between the power spectrum of the total intensity, the modal intensities, and the intensity phases. We shall then use the known phase properties of the AD regime in two specific examples to reduce these relations to closed relations between the power spectra of the modal and the total intensities.

Let us consider a dynamical system described by the evolution equation

$$\frac{dx}{dt} = f(\mathbf{x}, \xi), \tag{1}$$

where $\mathbf{x} = (\mathbf{I}, \mathbf{z}) = (I_1, I_2, \ldots, I_N, z_1, z_2, \ldots, z_L)$, $N + L = M$ are the variables of the system, and $\xi$ is the control parameter. The partition in $\mathbf{I}$ and $\mathbf{z}$ is arbitrary. The $\mathbf{I}$ variables are those whose power spectrum will be analyzed; the $\mathbf{z}$ variables are the remaining dynamical variables necessary to describe the system. For instance, in the examples we shall discuss below the $\mathbf{I}$ variables are the modal intensities and the $\mathbf{z}$ variables are the moments of the population inversion. We assume that there is a steady state $\mathbf{x}_{\mathrm{st}}$ which satisfies $f(\mathbf{x}_{\mathrm{st}}, \xi) = 0$. We define a deviation $\mathbf{U} = \mathbf{x} - \mathbf{x}_{\mathrm{st}} = (U_1, U_2, \ldots, U_M)$. Assuming that the nonlinearity in Eq. (1) is algebraic (in the physical examples, it will be quadratic), the linear stability of the steady state $\mathbf{x}$ will yield a set of $S \le M$ eigenvalues and $M$ eigenvectors. If we seek linearized solutions of the form $\mathbf{U} = \mathbf{V}\exp(\lambda t)$, we obtain eigenvalues of the form $\lambda_s = 2\pi i \Omega_s(\xi) - \kappa_s(\xi)$ with $\kappa_s(\xi) > 0$ in the domain where $\mathbf{x}_{\mathrm{st}}$ is stable. The solution of the linearized equation for $\mathbf{U}$ can, therefore, be written as

$$U_n(t, \xi) = \mathrm{Re}\left\{\sum_{s=1}^{S} A_{ns}(\xi) e^{i(2\pi\Omega_s t + \theta_{ns}) - \kappa_s t}\right\} \equiv \sum_{s=1}^{S} U_{ns}(t, \xi), \tag{2}$$

with real $A_{ns}$ and $\theta_{ns}$. The usual definition of the total power in the time interval $(0, T)$ of a component $f$ of $\mathbf{x}$ is

$$P(f, t) = \int_0^T |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(\Omega)|^2 \, d\Omega \equiv \int_{-\infty}^{\infty} P(f, \Omega) \, d\Omega \,. \tag{3}$$

This relation defines the power spectral density $P(f, \Omega)$ which is the central function we wish to study in terms of the Fourier transform $F(\Omega)$ of $f(t)$. The two models for which AD has been found are characterized by the property that the eigenvalues $\lambda_s$ depend on a large parameter $\mu \gg 1$ with the scaling

$$\mathrm{Im}(\lambda_s) \equiv 2\pi\Omega_s = \mathcal{O}(\mu^{1/2}), \quad \mathrm{Re}(\lambda_s) \equiv -\kappa_s = \mathcal{O}(1), \tag{4}$$

In this limit, and for $T \ge \mathcal{O}(1)$, we can approximate in the evaluation of the power spectral density

(PSD) the time integral over $\exp(-2\kappa_s t)[\cos(4\pi\Omega_s t + \theta_{rs} + \theta_{js}) + \cos(\theta_{rs} - \theta_{js})]$ by the time integral over $\exp(-2\kappa_s t)\cos(\theta_{rs} - \theta_{js})$. As a result, we obtain the following relation between the PSD of the modal variable $I_n$ and the PSD of the sum over all $I_n$ at the same frequency:

$$P(\Sigma I, \Omega_s) = \sum_{n=1}^{N} P(I_n, \Omega_s) + 2\sum_{r=1}^{N}\sum_{j=1}^{r-1}\sqrt{P(I_r, \Omega_s)P(I_j, \Omega_s)} \times \cos(\theta_{rs} - \theta_{js}). \quad (5)$$

The relation (5) still involves the initial condition via the phase difference in the cosine function. However, a signature of AD is precisely the occurrence of simple relations between the phases of the modal intensities. Therefore, we shall consider two models which are known to display AD and determine to what extent the relation (5) is simplified by these phase relations.

*(i) The Tang, Statz, and deMars rate equations [12].*—

$$\frac{dz_0}{dt} = w - z_0 - \sum_{n=1}^{N}\gamma_n\left(z_0 - \frac{z_n}{2}\right)I_n, \qquad \frac{dz_n}{dt} = \gamma_n z_0 I_n - z_n\left(1 + \sum_{k=1}^{N}\gamma_k I_k\right), \quad (6)$$

$$\frac{dI_n}{dt} = k\left[\gamma_n\left(z_0 - \frac{z_n}{2}\right) - 1\right]I_n, \qquad n = 1, 2, \ldots, N. \quad (7)$$

These equations describe the $N$-mode Fabry-Pérot laser with homogeneous broadening and include the influence of the population inversion grating $z_n$. The modal intensities are $I_n$ and the space average of the population inversion is $z_0$. The pump parameter $w$ is normalized so that the lasting threshold is $w = 1$ for the very first mode. The dimensionless decay rate $k$ is large for solid-state lasers [typical values range from $10^4$ for neodymium-doped yttrium aluminum garnet (Nd:YAG) lasers to $10^6$ for lithium neodymium tetraphospate (LNP) lasers]. The modes differ by the relative gain $\gamma_n \leq 1$ with $\gamma_1 \equiv 1$. The eigenvalues $\lambda_s$ and the eigenvectors $U_n$ of the linear stability analysis have been calculated explicitly in [13]. We have shown that the scaling (4) applies to the eigenvalues with $\mu = k$. If all the modes have the same gain, $\gamma_n = 1$, there is one low frequency $\Omega_L$ which is associated with $N - 1$ eigenvectors and a frequency $\Omega_R > \Omega_L$ which is nondegenerate. The total intensity satisfies the equation of a harmonic oscillator with frequency $\Omega_R$. Using this result, it is easy to obtain the following relations between the PSD of the modal and total intensities, $I_n$ and $\Sigma I$, respectively:

$$P(I_n, \Omega_R) \equiv P(I, \Omega_R), \qquad P(\Sigma I, \Omega_R) = N^2 P(I, \Omega_R), \qquad P(\Sigma I, \Omega_L) = 0. \quad (8)$$

The last relation is a signature of AD since it implies that the total intensity oscillates only at $\Omega_R$ which is the single mode frequency. Numerically, these relations are easily verified by a direct integration of Eqs. (6) and (7). For $k \geq 10^4$ we have recovered the result (8) within the numerical accuracy.

However, the relations (8) are of limited use due to the restriction that all modes have the same gain. In the case of arbitrary relative gains, there are $N$ different frequencies and the determination of the phases $\theta_{ns}$ requires the explicit solution of an algebraic equation of degree $N$. In the case of two modes this is easily done by generalizing to arbitrary $\gamma_2$ but $k \gg 1$, the calculation presented in Ref. [13]. This leads to phase differences which are either 0 (inphased oscillations) or $\pi$ (antiphased oscillations). As a result, we obtain for $N = 2$ but arbitrary relative gain

$$P(\Sigma I, \Omega_R) = \left(\sqrt{P(I_1, \Omega_R)} + \sqrt{P(I_2, \Omega_R)}\right)^2, \qquad P(\Sigma I, \Omega_L) = \left(\sqrt{P(I_1, \Omega_L)} + \sqrt{P(I_2, \Omega_L)}\right)^2. \quad (9)$$

Here again we have a result which is independent of the system preparation. To test these relations we have integrated numerically Eqs. (6) and (7) for $N = 2$, $w = 2.25$, and $k = 10^4$. The initial condition is the steady state with pump parameter $w = 2.5$. The power spectrum is calculated on the transient relaxation to the new steady state. Excellent agreement has been obtained between the results of the numerical simulation and the relations (9).

Up to this point, we have derived relations between PSD by introducing, step by step, the necessary simplifying assumptions. Let us now do the converse and ask what can be inferred from the fact that such relations are satisfied for a given system. For instance, we consider the two-mode relations (9). Their verification implies first and foremost that the perturbation applied to the system is weak enough to warrant a linear response. However, linearity implies only $P(I_1, \lambda_s) = |A|^2$, $P(I_2, \lambda_s) = |B|^2$, and $P(\Sigma I, \lambda_s) = |A + B|^2$. The coefficients $A$ and $B$ are real functions of $w$ and $i\lambda_s$. If the damping $(-\kappa_s)$ had the same scaling as $\Omega_s$, then $A$ and $B$ would be complex. Because of the scaling (4), $i\lambda_s$ is real to dominant order in $k$, and $A$ and $B$ are, therefore, also real (to the same dominant order in $k$), which then leads to $P(\Sigma I, \Omega_s) = (|A| \pm |B|)^2$ and hence to (9). Thus the verification of (9) also implies that the oscillations and the damping operate on very different time scales such as (4). This also indicates that the corrections to (9) will be $\mathcal{O}(\mu^{1/2})$.

An analysis of the reference model developed in Ref. [13] indicates that only the PSD at the same frequency may lead to universal relations. Relating PSD's at different frequencies always leads to relations which involve the initial condition.

*(ii) Intracavity second harmonic generation* [14].—We shall consider only the case where all modes oscillate with the same polarization. The evolution equations for the

modal intensities $I_n$ and the nonlinear gains $G_n$ are

$$\eta dI_n/dt = I_n\left(G_n - \alpha + g\varepsilon I_n - 2g\varepsilon \sum_{r=1}^{N} I_r\right), \quad (10)$$

$$dG_n/dt = \gamma - G_n\left(1 + (1-\beta)I_n + \beta \sum_{r=1}^{N} I_r\right), \quad (11)$$

where $\alpha$ is the cavity loss parameter, $\gamma$ is the small signal gain which is related to the pump rate, $\beta$ is the cross-saturation parameter, and $g$ is a geometrical factor whose value depends on the phase delays of the amplifying and doubling crystals and on the angle between the fast axes of these two crystals. We have assumed that $\alpha$, $\beta$, and $\gamma$ are mode independent, in good agreement with the experimental results [1,14]. The parameter $\eta$ equals $\tau_c/\tau_f$, where $\tau_c$ and $\tau_f$ are the cavity round trip time and fluorescence lifetime, respectively. The experiments have been performed in the parameter domain $\varepsilon \ll 1$, $\eta \ll 1$ with $\alpha, \beta, \gamma, g, \varepsilon/\eta = \mathcal{O}(1)$. In Ref. [15], we have shown that the linear stability of the steady state $I_n = I$, $G_n = G$ leads to only two eigenvalues $\lambda_{1,2} = 2\pi i \Omega_{1,2} - \kappa_{1,2}$ which satisfy the scaling relations (4) with $\mu = 1/\eta$. The two frequencies are $2\pi\Omega_1 = \sqrt{(1-\beta)IG/\eta} + \mathcal{O}(1)$ and $2\pi\Omega_2 = \sqrt{[1+(N-1)\beta]IG/\eta} + \mathcal{O}(1)$. Here again the total intensity satisfies the equation of a harmonic oscillator with frequency $\Omega_2$. The eigenvectors of the linearized problem have been determined analytically. With these results, we can compute the phases $\theta_{ns}$ which appear in (5). For instance, for two modes we recover the relations (9) while for three modes we have

$$P(\Sigma I, \Omega_2) = \left(\sum_{n=1}^{3} \sqrt{P(I_n, \Omega_2)}\right)^2. \quad (12)$$

The generalization of the relation at $\Omega_2$ for an arbitrary number of modes is obvious. For $\Omega_1$, the situation is more difficult. In the same way as we have shown analytically that there are two different periodic solutions which differ only by their basin of attraction [16], the linear stability of the steady state also indicates that two different sets of phase relations can occur for $N = 3$, each leading to a different relation between $P(\Sigma I, \Omega_1)$ and the $P(I_n, \Omega_1)$. The situation described in Table I corresponds to $\theta_{31} - \theta_{11} = \theta_{31} - \theta_{21} = \pi$ at $\Omega_1$ and $\theta_{32} = \theta_{12} = \theta_{22}$ at $\Omega_2$. It corresponds to the regime called AD2 in Ref. [16] though with damping in this case. The other possibility is $\theta_{31} = 0$, $\theta_{21} = 2\pi/3$, and $\theta_{11} = 4\pi/3$ at $\Omega_2$ and $\theta_{32} = \theta_{12} = \theta_{22}$ at $\Omega_1$. This corresponds to the regime labeled AD1 in Ref. [16] apart from the damping present in this problem.

We have made a double test of the relations (5). First, we have considered the relaxation to the stable-steady state when the initial condition is a small perturbation of that steady state. The results are displayed in the first row of data in Table I for three modes. For the initial conditions given in Table I, the phase differences are 0 or $\pi$ and the relation (5) becomes

$$P(\Sigma I, \Omega_1) = \sum_{p=1}^{3} P(I_p, \Omega_1) + 2\sqrt{P(I_1,\Omega_1)P(I_2,\Omega_1)} - 2\sqrt{P(I_3,\Omega_1)} \times \left(\sqrt{P(I_2,\Omega_1)} + \sqrt{P(I_1,\Omega_1)}\right). \quad (13)$$

The agreement between the numerical integration of (10) and (11) and the relations (12) and (13) is striking. This is essentially due to the smallness of $\varepsilon$ and $\eta$ for which we have chosen values that are given by the experimental

TABLE I. PSD for intracavity second harmonic generation with three modes. Parameters are $g = 0.9$, $\alpha = 0.01$, $\beta = 0.6$, $\gamma = 0.05$, $\varepsilon = 10^{-6}$, and $\eta = 2\varepsilon$. Initial condition: $I_1 = 1.81$, $I_2 = 1.805$, $I_3 = 1.79$, $G_1 = 0.010\,008\,2$, $G_2 = 0.010\,008\,2$, and $G_3 = 0.010\,008\,2$. Relaxation frequencies: $\Omega_1 = 9.6$ and $\Omega_2 = 22.5$. The variable $\varphi_{rjs}$ stands for $\theta_{rs} - \theta_{js}$. The PSD are normalized to $P(\Sigma I_n, \Omega_2)$.

| $\delta$ | $\Omega$ | $\Omega_s$ | $\varphi_{31s}$ | $\varphi_{32s}$ | $\varphi_{21s}$ | $P(I_1,\Omega_s)$ | $P(I_2,\Omega_s)$ | $P(I_3,\Omega_s)$ | $P(\Sigma I_n,\Omega_s)$ Numerical | $P(\Sigma I_n,\Omega_s)$ Calculated |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | $\Omega_1$ | $\pi$ | $\pi$ | 0 | 0.281 | 0.046 | 0.549 | <0.001 | <0.001 |
| | | $\Omega_2$ | 0 | 0 | 0 | 0.112 | 0.111 | 0.111 | 1 | 1 |
| 0.001 | 5 | $\Omega$ | 0 | 0 | 0 | 0.256 | 0.256 | 0.256 | 2.303 | 2.303 |
| | | $\Omega_1$ | $\pi$ | $\pi$ | 0 | 0.208 | 0.033 | 0.410 | <0.001 | <0.001 |
| | | $\Omega_2$ | 0 | 0 | 0 | 0.111 | 0.111 | 0.111 | 1 | 1 |
| 0.001 | $\Omega_1$ | $\Omega_1$ | — | — | — | 1.063 | 0.610 | 1.147 | 3.487 | — |
| | | $\Omega_2$ | 0 | 0 | 0 | 0.111 | 0.111 | 0.111 | 1 | 1 |
| 0.001 | 15 | $\Omega_1$ | $\pi$ | $\pi$ | 0 | 0.175 | 0.028 | 0.346 | <0.001 | <0.001 |
| | | $\Omega$ | 0 | 0 | 0 | 0.789 | 0.791 | 0.799 | 7.136 | 7.136 |
| | | $\Omega_2$ | 0 | 0 | 0 | 0.111 | 0.111 | 0.111 | 1 | 1 |
| 0.001 | $\Omega_2$ | $\Omega_1$ | $\pi$ | $\pi$ | 0 | <0.001 | <0.001 | <0.001 | <0.001 | <0.001 |
| | | $\Omega_2$ | 0 | 0 | 0 | 0.111 | 0.111 | 0.111 | 1 | 1 |
| 0.001 | 30 | $\Omega_1$ | $\pi$ | $\pi$ | 0 | 0.409 | 0.066 | 0.798 | <0.001 | <0.001 |
| | | $\Omega_2$ | 0 | 0 | 0 | 0.112 | 0.111 | 0.110 | 1 | 1 |
| | | $\Omega$ | 0 | 0 | 0 | 0.674 | 0.673 | 0.672 | 6.055 | 6.055 |

data. Using the eigenvectors of the linearized problem one can show analytically that $P(\Sigma I, \Omega_1)/P(I_n, \Omega_1) = \mathcal{O}(\eta)$, $n = 1, 2, \ldots, N$. The second test we have performed is to integrate Eqs. (10) and (11) starting with the same initial condition but adding a weak modulation of the gain $\gamma$ which becomes $\gamma[1 + \delta \cos(2\pi\Omega t)]$ with $\delta \ll 1$. The phases $\theta_{ns}$ have been determined by inspection of the solutions and we again observe an excellent agreement between the PSD of the total intensity and the PSD of the modal intensities as given by (12) and (13), save for the case $\Omega = \Omega_1$ where the relative phases could not be determined. This is because the modulation acts with the same phase on all the modes. At $\Omega_2$, the modes are in phase while at $\Omega_1$ they are in antiphase. Because of this, we cannot determine the relative phase of the driven modes. Another way to state the same problem is that even in the case of perfect AD there will be a residual peak at $P(\Sigma I, \Omega_1)$, which is due to the modulation and not to the internal dynamics.

Stimulating discussions with T. Erneux and D. Pieroux are gratefully acknowledged. This research was supported in part by the Fonds National de la Recherche Scientifique (Belgium) and the Inter-University Attraction Pole program of the Belgian government.

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