ac Josephson Effect in a Single Quantum Channel

D. Averin and A. Bardas
Department of Physics, State University of New York at Stony Brook, New York, New York, 11794
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We have calculated all the components of the current in a short one-dimensional channel between two superconductors for arbitrary voltages and transparencies D of the channel. We demonstrate that in the ballistic limit ($D \simeq 1$) the crossover between the quasistationary evolution of the Josephson phase difference φ at small voltages and transport by multiple Andreev reflections at larger voltages can be described as the Landau-Zener transition induced by finite reflection in the channel. For perfect transmission and vanishing energy relaxation rates the stationary current-phase relation is never recovered, and $I(\varphi) = I_c \mid \sin\varphi/2 \mid \operatorname{sgn} V$ for arbitrary small voltages.

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It has been known for more than 20 years that electron transport in short superconducting weak links with arbitrary transparency can be described in terms of multiple Andreev reflections (MAR) [1]. Despite this, a quantitative understanding of the ac Josephson effect in these structures is still not complete. Various approaches to quantitative calculations of the current at finite voltages [2–5] were mainly focused on the dc current which exhibits the so-called subharmonic structure, i.e., current singularities at voltages $V = 2\Delta/en$, $n = 1, 2, \ldots$, where Δ is the superconducting energy gap—see, e.g., [6,7], and references therein. However, a dc current carries only indirect information about weak link dynamics, whereas calculations of the ac currents [8,9] have been limited to large voltages; the limitation being caused by the fact that at small voltages it is necessary to take into account an increasingly large number of Andreev reflections.

The aim of our work was to study a model of a short constriction between two superconductors, which permits the quantitative description of the current dynamics for arbitrary voltages and transparencies of the constriction. In this model, we found a new regime in the constriction dynamics, which occurs at small voltages and connects quasistatic variations of the Josephson phase difference at $V \rightarrow 0$ with MAR at larger voltages.

We consider a single-mode channel of electron gas with transparency D between two superconductors (the calculations can be generalized in a straightforward way to several separable modes). The length d of the channel is assumed to be much smaller than the coherence length ξ as well as the elastic and inelastic scattering lengths in the superconductors. This allows us to neglect scattering in the vicinity of the channel (besides that described by the reflection probability $R = 1 - D$) and makes it convenient to describe electron motion in the constriction with the time-dependent Bogolyubov–de Gennes (BdG) equations. Assuming that the Fermi energy in the constriction is much larger than the energy gap Δ, we simplify these equations further by adopting the quasiclassical approximation. The condition $d \ll \xi$ makes the superconducting properties of the constriction itself irrelevant (even if there is a finite Δ in the constriction we can neglect it in the BdG equations on the small space scale given by d) [10]. It is easier to visualize electron motion in the channel assuming that the constriction is normal ($\Delta = 0$), so that the transport through the resulting superconductor–normal-metal–superconductor (SNS) structure can be described directly in terms of the Andreev reflection at the two NS interfaces. We adhere to this framework in what follows.

The final simplification is that impedance of the single- (or few-)mode channel is on the order of h/e^2 and is much larger than the characteristic impedance of a typical external circuit. This eliminates the necessity (essential for a realistic description of the Josephson junctions with low resistance) of determining the dynamics of the Josephson phase difference and voltage across the channel self-consistently. We assume that the voltage is constant in time.

The model we obtain is directly applicable to the atomic-size Josephson junctions [11] which exhibit ballistic quantization of the stationary critical current [12]. Another context of current interest in which the model is relevant is high-critical-current Josephson junctions [6,13], which are believed to be adequately represented as an ensemble of atomic-size microconstrictions, each of which carries a few conducting electron modes.

Under the assumptions outlined above, the BdG equations for transport in the constriction can be solved in terms of the two scattering processes for electrons and holes, along the same lines as in the stationary case [14]. One process is the Andreev reflection at the NS interfaces characterized by the reflection amplitude a as a function of the quasiparticle energy ϵ:

$$a(\epsilon) = \frac{1}{\Delta} \times \begin{cases} \epsilon - \operatorname{sgn}(\epsilon)(\epsilon^2 - \Delta^2)^{1/2}, & \mid \epsilon \mid > \Delta, \\ \epsilon - i(\Delta^2 - \epsilon^2)^{1/2}, & \mid \epsilon \mid < \Delta. \end{cases} \tag{1}$$

Another process is electron scattering in the constriction characterized by a scattering matrix

$$S_{el} = \begin{pmatrix} r & t \\ t & -r^* t/t^* \end{pmatrix}, \tag{2}$$

where $|t|^2 = D$ and $|r|^2 = R$. The scattering matrix for holes is the time reverse of S_{el}, $S_h = S_{el}^*$.

The last ingredient of the scattering scheme is the fact that the energy of an electron is increased by eV each time it passes through the channel from left to right, while the hole increases its energy passing through the constriction in the opposite direction. Because of this, the electron and hole wave functions are sums of the components with different energies shifted by $2eV$. For instance, the wave functions in region I (Fig. 1) generated by the quasiparticle incident from the left superconductor onto the channel can be written as

$$\psi_{el} = \sum_n [(a_{2n}A_n + J\delta_{n0})e^{ikx} + B_n e^{-ikx}]e^{-i(\epsilon+2neV)t/\hbar},$$
$$\psi_h = \sum_n [A_n e^{ikx} + a_{2n}B_n e^{-ikx}]e^{-i(\epsilon+2neV)t/\hbar}, \tag{3}$$

where k and ϵ are momentum (equal to the Fermi momentum) and energy of the incident quasiparticle, and $a_m \equiv a(\epsilon + meV)$. In Eq. (3) we took into account the fact that the amplitudes of electron and hole waves are related by the Andreev reflection and that the quasiparticle incident from the superconductor produces an electron in the normal region with amplitude $J(\epsilon) = [1 - |a(\epsilon)|^2]^{1/2}$. The wave function in region II has a similar form with two modifications: it does not have the source term J and is shifted in energy by eV.

The wave amplitudes in regions I and II are related by the scattering matrix (2):

$$\begin{pmatrix} B_n \\ C_n \end{pmatrix} = S_{el}\begin{pmatrix} a_{2n}A_n + J\delta_{n0} \\ a_{2n+1}D_n \end{pmatrix},$$
$$\begin{pmatrix} A_n \\ D_{n-1} \end{pmatrix} = S_h\begin{pmatrix} a_{2n}B_n \\ a_{2n-1}C_{n-1} \end{pmatrix}. \tag{4}$$

Eliminating the wave amplitudes C_n and D_n in region II from Eq. (4), we obtain the recurrence relation for the amplitudes A_n and B_n:

$$D\frac{a_{2n+2}a_{2n+1}}{1 - a_{2n+1}^2}B_{n+1} - \left[D\left(\frac{a_{2n+1}^2}{1 - a_{2n+1}^2} + \frac{a_{2n}^2}{1 - a_{2n-1}^2}\right) + 1 - a_{2n}^2\right]B_n + D\frac{a_{2n}a_{2n-1}}{1 - a_{2n-1}^2}B_{n-1} = -R^{1/2}J\delta_{n0},$$
$$A_{n+1} - a_{2n+1}a_{2n}A_n = R^{1/2}(B_{n+1}a_{2n+2} - B_n a_{2n+1}) + Ja_1\delta_{n0}. \tag{5}$$

These recurrence relations can be solved with the method developed in Refs. [5,15]. The amplitudes A_n and B_n of the wave functions (3) obtained in this way determine all Fourier components of the current $I(t)$ in the channel:

$$I(t) = \sum_k I_k e^{i2keVt/\hbar}.$$

Collecting contributions from the quasiparticles incident on the channel from the two superconductors and making use of the fact that $A(-\epsilon, -V) = -A^*(\epsilon, V)$ and $B(-\epsilon, -V) = B^*(\epsilon, V)$ [as follows from the recurrence relations (5) and the form of the Andreev reflection amplitude (1)] we finally arrive at

$$I_k = \frac{e}{\pi\hbar}\Bigg[eV\delta_{k0} - \int d\epsilon\, \tanh\left\{\frac{\epsilon}{2T}\right\} \times \Big(J(\epsilon)(a_{2k}A_k^* + a_{-2k}A_{-k}) + \sum_n (1 + a_{2n}a_{2(n+k)}^*)(A_n A_{n+k}^* - B_n B_{n+k}^*)\Big)\Bigg]. \tag{6}$$

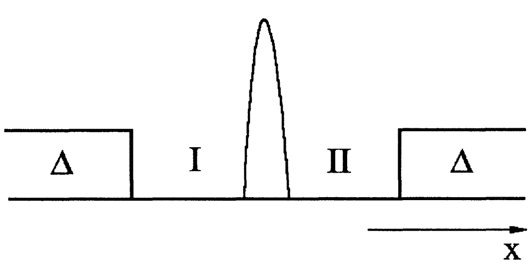

FIG. 1. A schematic energy diagram of a short one-dimensional channel with arbitrary transparency between two superconductors. I and II denote the portions of the channel separated by the scattering region.

Some results of the numerical calculations of the current from the recurrence relations (5) and equations (6) are shown in Fig. 2. One can see that the ac components of the current exhibit the subgap singularities at $V = 2\Delta/n$ similar to those in the dc current. It is clear from Fig. 2 (and straightforward to show analytically) that in the limit $D \rightarrow 0$ Eqs. (5) and (6) reproduce the tunnel Hamiltonian expressions for the current. In the case of zero temperature (shown in Fig. 2) the only component of the current that is nonvanishing at finite reflection probability and small voltages is the stationary Josephson current. In particular, it can be checked that the limiting ($V = 0$) values of the sine component [Fig. 2(b)] coincide with the first Fourier harmonics of the stationary Josephson current $I(\varphi) = (eD\Delta/2\hbar)\sin\varphi/[1 - D\sin^2(\varphi/2)]^{1/2}$.

Figure 2(c) shows that for large transmission probabilities D the cosine component of the current is negative in the wide range of voltages below the gap voltage $2\Delta/e$. [Calculations based on Eqs. (5) and (6) at finite temperatures show that this feature is preserved at temperatures up to about 0.3Δ.] This fact provides a possible resolution of the long-standing problem of the sign of the cosine component of the ac Josephson current in tunnel junctions (see, e.g., [16]), if one assumes that due to nonuniformity of tunnel barriers realistic tunnel junctions always contain regions with high transparency.

The feature of the curves in Fig. 2, which to our knowledge has never before been discussed, is the rapid variation of all current components at small voltages and small reflection probabilities. In order to understand this new feature we consider first the case of perfect

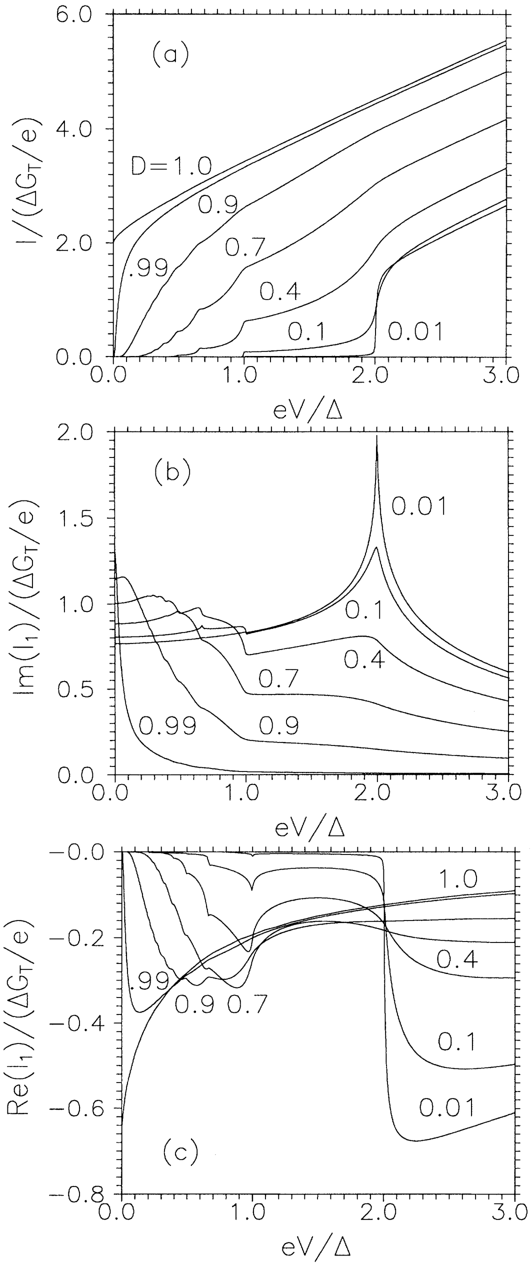

FIG. 2. The dc current and the sine and cosine parts of the first Fourier component of the ac current in a short quantum channel between the two superconductors at zero temperature. G_T is the normal-state conductance of the channel $G_T = e^2 D/\pi\hbar$. All curves exhibit subharmonic singularities at $V = 2\Delta/n$ associated with multiple Andreev reflections. Rapid variation of the curves with $D \simeq 1$ at small voltages is a manifestation of the Landau-Zener transitions between the quasistationary current-carrying states in the energy gap. For details see text.

transmission, $D = 1$. In this case the recurrence relations can be solved explicitly,

$$B_n = 0, \quad A_n = J \prod_{j=1}^{2n-1} a_j \,. \tag{7}$$

This solution shows that in the limit $V \rightarrow 0$, when electrons and holes increase, their energy in the Andreev reflections cycles very slowly; the amplitudes A_n decrease very rapidly as functions of energy outside the energy gap, since $|a(\epsilon)| < 1$ at these energies. At the same time, the number of Andreev reflections inside the gap increases. This means that in this regime the dominant contribution to the current comes from the gap interval $|\epsilon| < \Delta$, where A_n are not decaying [$a(\epsilon) = \exp\{-i \arccos(\epsilon/\Delta)\}$, $|a| = 1$) and the number of Andreev reflection cycles diverges as Δ/eV. This divergence is regularized by the fact that the particles getting into the energy gap originate only from a small energy range near the gap edges. With this understanding, Eq. (6) for the kth Fourier component of the current can be simplified for $V \ll \Delta/e$ as follows:

$$I_k = \frac{e\Delta}{\pi\hbar} \tanh\Big(\frac{\Delta}{2T}\Big) \int_{-1}^{1} dz \exp\{2ik \arccos z\}\,. \tag{8}$$

Equation (8) means that the current-phase relation at finite voltages is

$$I(\varphi) = I_c \,|\sin\varphi/2|\, \mathrm{sgn} V\,, \quad I_c = \frac{e\Delta}{\hbar} \tanh\Big(\frac{\Delta}{2T}\Big), \quad \varphi = 2eVt/\hbar\,; \tag{9}$$

in particular, the dc current is $2I_c/\pi$, the sine part of the first Fourier component of the ac current is vanishing, while the cosine part is $-2I_c/3\pi$.

Expression (9) has a natural interpretation in terms of the quasistationary discrete states inside the gap that are responsible for the stationary Josephson current [12,17]. Two such states with energies $E_\pm = \Delta\cos\varphi/2$ (Fig. 3) carry, respectively, forward and backward currents, and in the stationary case ($\varphi =$ const) are occupied according to the equilibrium Boltzmann distribution. At finite voltage the energy of these states is changing in time due to the evolution of φ. For the vanishing inelastic scattering rate the density of states in the superconductors is also vanishing within the gap, so that the occupation probabilities of the two current-carrying states remain constant as long as they are moving inside the energy gap $E_\pm < \Delta$. The only point at which the occupation probabilities can change is at the gap edges $\epsilon = \pm\Delta$ (i.e., at $\varphi = 2\pi n$). These considerations immediately give Eq. (9).

Solution (7) of the scattering problem at $D = 1$ gives some analytical results at larger voltages also. In particular, it can be shown from Eqs. (6) and (7) that at $T = 0$ the sine part of the first Fourier component of the ac Josephson current vanishes identically at all voltages, while the cosine part approaches the asymptote $-I_c(\Delta \ln 2/2eV)$ at $eV \gg \Delta$. Both of these results agree with the numerical results shown in Fig. 2.

The reasoning that lead to Eq. (9) can easily be generalized to the small but finite reflection probability R of the constriction. Finite reflection creates a finite matrix element r of the transition between two current-carrying states in the energy gap which occur near the point $\varphi = \pi$, where the energies $E_\pm$ of these states coincide—see Fig. 3. The problem of this transition is then a standard level-crossing problem, and the probability p that the system will continue to occupy the same level after crossing the point $\varphi = \pi$ is given by the Landau-Zener expression. In our notation this expression is

$$p = \exp\left\{-\frac{\pi R \Delta}{eV}\right\}. \tag{10}$$

The finite transition probability p modifies the current-phase relation (9) as follows:

$$I(\varphi) = I_c \mathrm{sgn} V \times \begin{cases} \sin\varphi/2, & 0 < \varphi < \pi, \\ (2p-1)\sin\varphi/2, & \pi < \varphi < 2\pi. \end{cases} \tag{11}$$

In the relevant range of parameters (small R and V), Eq. (11) reproduces the result of the numerical solution of the recurrence relations (5). Indeed, at very small V ($eV < R\Delta$), $p \to 0$ and the system in its evolution follows a current-phase relation which at $T = 0$ coincides with the stationary relation $I = I_c \sin\varphi/2\, \mathrm{sgn}\,(\cos\varphi/2)$. As a result, the dc current and the cosine component of the first harmonic of the ac current are vanishing, while the sine component is equal to its stationary value, in agreement with Fig. 2. At larger voltages ($eV > R\Delta$), $p \to 1$ and the current-phase relation approaches the one for $D = 1$, $I(\varphi) = I_c \mathrm{sgn} V \mid \sin\varphi/2 \mid$. For this $I(\varphi)$ the sine component is zero, while the cosine component and dc current are nonvanishing. All this means that the reason for the rapid variation of all current components with voltage at $D \simeq 1$ and small voltages (see Fig. 2) is that the probability (10) of the Landau-Zener transition between the two current-carrying states changes rapidly on a small voltage scale given by $R\Delta$.

Before concluding, we would like to mention that our calculations agree with most of the previous results on the ac Josephson effect in short constrictions, in the parameter ranges where previous results are available. In particular, our numerical results (Fig. 2) agree with the numerical results of Arnold [9] for large voltages. At $D = 1$, Eq. (9) of our work gives the dc current in agreement with that obtained by Gunsenheimer and Zaikin [4]. There is, however, a contradiction between calculations at large voltages based on the solution (7) and Zaitzev's results for ac components of the current at $D = 1$ and large voltages [8]. The reason for this contradiction is not clear at present.

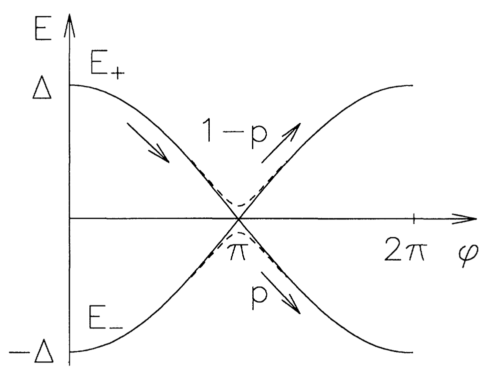

FIG. 3. The energy diagram of the two quasistationary current-carrying states $E_\pm$ in the constriction. The arrows show two possible routes of the system evolution due to Landau-Zener transitions in the vicinity of the level-crossing point $\varphi = \pi$.

In conclusion, we have calculated the current in a short single-mode electron channel between two superconductors for arbitrary voltages and transparencies of the channel. To the best of our knowledge this is the first time a full description of the current dynamics in a weak link has been developed. In the ballistic limit $D \simeq 1$, crossover from quasistationary Josephson current at smaller voltages to multiple Andreev reflections at larger voltages occurs at $V \simeq \pi R \Delta / e$ and can be described in terms of Landau-Zener tunneling between the discrete current-carrying states in the energy gap which are responsible for the stationary Josephson current at $V = 0$.

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