Driven Interfaces with Phase Disorder

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(Received 2 March 1995)

A variety of systems including charge density waves, flux line arrays, and surfaces of disordered crystals can be described by the driven, phase-disordered sine-Gordon equation. Here it is shown that the dominant effect of the phase disorder in the uniformly driven state is to introduce a quenched random mobility for the moving "interface" or "phase" variable. Analytic predictions are obtained by mapping the resulting disordered Kardar-Parisi-Zhang equation to a directed polymer problem, and the predictions are compared to simulations of one-dimensional phase-disordered solid-on-solid models.

PACS numbers: 61.50.Cj, 71.45.Lr, 74.60.Ge

Several condensed matter systems of current interest can be described in terms of a phaselike variable $h(\mathbf{x}, t)$ that evolves subject to the conflicting influences of a uniform driving force, a harmonic spatial coupling, and a pinning potential that seeks to enforce a preferred local random phase $h = \phi(\mathbf{x})$. The main examples [1] are (i) charge density waves (CDW's) [2,3], (ii) arrays of flux lines in dirty type-II superconductors [4], and (iii) crystalline films growing on a disordered substrate [5,6]. These systems are commonly modeled by the phase-disordered, driven, noisy sine-Gordon equation

$$\partial h/\partial t = \gamma \nabla^2 h - V_0 \sin 2\pi [h(\mathbf{x}, t) - \phi(\mathbf{x})] + F + \eta(\mathbf{x}, t). \tag{1}$$

Here γ is the stiffness, V_0 the strength of the pinning potential, F the driving force, and η is thermal noise with short-range correlations in time and space.

In the first two examples given above h is a periodic variable defined on the unit interval, describing the CDW phase and the deviations of the flux line array from the ordered state, respectively, while in example (iii) it is an unbounded quantity measuring the height of the crystal surface above the substrate. However, in the absence of dislocations—which is, anyhow, a necessary condition for the applicability of (1) to problems (i) and (ii)—one may ignore the periodicity of h and use an extended zone scheme where the unit circle is "unwrapped" along the real line. Thus in the following we shall regard $h(\mathbf{x}, t)$ as unbounded and refer to it as the "interface height" above a point $\mathbf{x}$ in the d-dimensional substrate plane.

Recent work [7] has focused on the equilibrium ($F = 0$) behavior of (1) in $d = 2$, with the hope of gaining insight into the nature of the vortex glass phase in two-dimensional superconducting films [4]. While the existence of a glassy low temperature phase seems to be firmly established, different analytic methods have yielded conflicting predictions regarding its properties; numerical approaches have so far remained inconclusive [8,9].

In this Letter I address the dynamics of (1) at finite driving force $F > 0$. The primary goal will be to obtain a large scale description (beyond the vertical lattice spacing of unity) of the height fluctuations in the moving state. To put this issue into perspective, recall that in the *absence* of phase disorder ($\phi = \text{const}$) the fluctuations of the moving interface have been shown to be governed by the Kardar-Parisi-Zhang (KPZ) equation [10], both in $d = 1$ [11] and in $d = 2$ [12]. Here I will show that the most important effect of phase disorder is to induce a *spatially random, frozen contribution to the coarse-grained interface mobility* [13], leading to large scale fluctuations described by a KPZ-type equation with quenched random growth rates,

$$\partial h/\partial t = v_0 + \gamma \nabla^2 h + \mathbf{c}(\mathbf{x}) \cdot \nabla h + \tfrac{1}{2}[\lambda_0 + \lambda_1(\mathbf{x})](\nabla h)^2 + \eta(\mathbf{x}, t) + \epsilon(\mathbf{x}). \tag{2}$$

In addition to the additive random force $\epsilon(\mathbf{x})$, the random mobility also induces a random contribution to the coefficient $\lambda = \lambda_0 + \lambda_1$ of the KPZ nonlinearity $(\nabla h)^2$ and a random lateral drift velocity $\mathbf{c}(\mathbf{x})$. Provided the average KPZ coupling λ_0 is nonzero, the spatially varying contributions λ_1 and $\mathbf{c}$ are irrelevant by power counting and can be neglected. Via the well-known Cole-Hopf transformation [10] Eq. (2) can then be mapped onto a problem of directed paths in a random potential consisting of point and columnar defects, for which analytic predictions are available [14]. However, if $\lambda_0 = 0$, (2) describes a novel universality class. Power counting indicates that the tilt-dependent terms $\mathbf{c} \cdot \nabla h$ and $\frac{1}{2}\lambda_1(\nabla h)^2$ are relevant in dimensions $d < 2$. I will argue below that the case $\lambda_0 = 0$, while probably not generic for the three physical problems described above, is in fact realized by the driven sine-Gordon equation (1).

Rather than working with Eq. (1), I use a class of discrete, phase-disordered solid-on-solid (SOS) models [9,15] which can be regarded to arise from the strong coupling ($V_0 \to \infty$) limit of (1). For large V_0 the height is forced to assume the discrete set of values

$$h_{\mathbf{x}} = n_{\mathbf{x}} + \phi_{\mathbf{x}}, \tag{3}$$

corresponding to the minima of pinning potential, where $n_{\mathbf{x}}$ is an integer. At the same time the substrate space has

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to be discretized, so $\mathbf{x}$ becomes a site on a d-dimensional lattice. The harmonic coupling is introduced through the Hamiltonian

$$\mathcal{H} = \sum_{\langle \mathbf{xy} \rangle} |h_\mathbf{x} - h_\mathbf{y}|^q, \qquad (4)$$

where the sum runs over nearest neighbor pairs; $q = 1$ and $q = 2$ correspond to the standard SOS model and the discrete Gaussian model, respectively. Driven dynamics is implemented by Metropolis transition rates

$$\mathcal{R}[h_\mathbf{x} \rightarrow h_\mathbf{x} \pm 1] = \min[1, \exp(-\Delta \mathcal{H}_\pm / k_B T)], \qquad (5)$$

where $\mathcal{H}_\pm = \mathcal{H}_{\mathrm{fin}} - \mathcal{H}_{\mathrm{in}} \mp F$ is the energy change associated with the move, supplemented with a bias contribution F; for $F > 0$ the heights will steadily increase.

To see how a random mobility arises in this model, consider first a local region where the substrate is almost flat, i.e., $\phi_\mathbf{x} \approx \phi_0$ for all $\mathbf{x}$ [Fig. 1(a)]. At low temperatures and small driving force the interface velocity will then be exponentially small; more precisely, if we assume that the region can be approximated by a perfect singular substrate, the mobility vanishers exponentially for $(k_B T)^{-1} \rightarrow \infty$ in $d = 1$, and it is identically zero below the roughening temperature in $d = 2$ [16]. Next, consider a region where the random phases are close to perfectly staggered, i.e., an alternating pattern $\phi_i \approx \phi_0 + \frac{1}{4}(-1)^i$ in $d = 1$, or "checkerboard" $\phi_{ij} \approx \phi_0 + \frac{1}{4}(-1)^{i+j}$ in $d = 2$ [Fig. 1(b)]. In such a region it is possible to add particles to the crystal indefinitely without expending energy, and consequently the mobility remains nonzero for $T \rightarrow 0$; in fact, the zero temperature limit of the SOS model on a staggered substrate is the well-known single step growth model [17].

We conclude that at low temperatures different regions of the interface will have widely different mobilities, depending on the local configurations of the random phases. By the same token, the dependence of the mobility on the local inclination of the interface [10–12,18] will vary in space. For example, for the flat configuration in Fig. 1(a) the mobility is a strongly increasing function of inclination at low temperatures, while in the staggered case [Fig. 1(b)] it decreases with inclination [11,18]; a contribution to the mobility that is linear in ∇h arises from phase configurations that break the $x \rightarrow -x$ symmetry [Fig. 1(c)]. These considerations can be summarized in an expansion for the mobility of the form $\mu = \overline{\mu} + \mu_0(\mathbf{x}) + \mu_1(\mathbf{x}) \cdot \nabla h + \frac{1}{2}[\overline{\mu}_2 + \mu_2(\mathbf{x})](\nabla h)^2 +$ higher order terms. Writing the local interface velocity as $\partial h / \partial t = \mu F$, we obtain (2) with $v_0 = \overline{\mu} F$, $\mathbf{c} = F \boldsymbol{\mu}_1$, $\lambda = \mu_2 F$, and $\epsilon = \mu_0 F$. By construction the random growth rates $\epsilon(\mathbf{x})$ (as well as the other random coefficients) in (2) are *bounded*; i.e., their distribution has a finite support, being limited by the mobilities of the fastest and slowest local phase configurations.

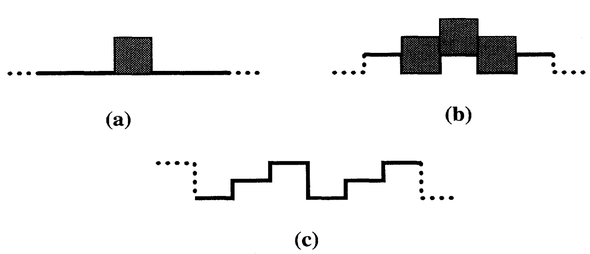

FIG. 1. Simple phase configurations of the one-dimensional phase-disordered SOS model. Deposition (indicated by shaded particles) on a flat surface (a) requires a finite amount of energy, while on the staggered substrate (b) growth can proceed indefinitely without energy cost; (c) is an example of an asymmetric configuration.

The argument shows that the disorder has a *persistent* influence on the moving interface, which accumulates over time. This effect was discussed in the CDW context by Coppersmith, who argued that it will in fact lead to phase slips and a breakdown of the description by (1) in sufficiently low dimensionalities [3]. In essence, she analyzed the large scale equation (2) in the absence of the tilt-dependent terms $\mathbf{c} \cdot \nabla h$ and $(\lambda/2)(\nabla h)^2$. This linear problem is easily solved, and one finds indeed that the local "phase" gradient diverges for $d \leq 2$ as $\langle (\nabla h)^2 \rangle \sim t^{(2-d)/2}$ with time or as $\langle (\nabla h)^2 \rangle \sim L^{2-d}$ with length scale. However, in the presence of inclination-dependent terms the situation changes profoundly, because then the interface can locally change its velocity by assuming a nonzero tilt; in this way different parts of the system can adapt to a common speed, and the *local* gradients remain bounded at the expense of an increase in the *long wavelength* fluctuations.

A simple manifestation of this mechanism was pointed out recently by Balents and Fisher [19], who studied the linearized equation (2) in the presence of a *constant* drift velocity $\overline{\mathbf{c}}$ and showed that such a term implies bounded phase gradients even in $d = 1$. A nonzero average drift arises naturally in the context of sliding CDW's [19] but is excluded by symmetry in the case of (untilted) interfaces of primary interest here.

To explore the dynamical consequences of the random mobility I have carried out simulations of two one-dimensional phase-disordered SOS models, governed by (4) with $q = 1$ and $q = 2$. Before discussing the results, I summarize the properties of (2) for $\lambda_0 \neq 0$. Neglecting the random contributions $\mathbf{c}(\mathbf{x})$ and $\lambda_1(\mathbf{x})$, the height function $h(\mathbf{x}, t)$ can be viewed [10,20] as the restricted free energy of a directed, flexible line (a "directed polymer") of length t, with one end fixed at the point $(\mathbf{x}, t)$ in $(d + 1)$-dimensional "space-time" and subject to quenched point $[\eta(\mathbf{x}, t)]$ and columnar $[\epsilon(\mathbf{x})]$ disorder, the latter having a distribution of finite support. Flory-type arguments applied to this problem yield the following predictions [14]: Starting from a flat configuration $h =$ const at time $t = 0$, typical configurations at time t show a hill-valley structure with the distance between peaks increasing as

$$\xi(t) \sim t / (\ln t)^\psi, \qquad (6)$$

where $\psi = 2$ in $d = 1$ and $\psi \approx 1.60$ in $d = 2$. The slope of the hillsides asymptotically (but slowly) approaches a fixed finite value. Consequently, the interface width $W = \langle (h - \langle h \rangle)^2 \rangle^{1/2}$ becomes proportional to ξ, and the height difference correlation function behaves as

$$G(r,t) = \langle [h(\mathbf{x} + \mathbf{r}, t) - h(\mathbf{x}, t)]^2 \rangle \sim r^2 \qquad (7)$$

for $r < \xi$. As the slopes of the hillsides steepen, the average interface velocity (the free energy per unit length of the polymer) $v(t) = \langle \partial h / \partial t \rangle$ approaches its asymptotic value $v(\infty)$ with a leading logarithmic correction

$$\Delta v = v(t) - v(\infty) \sim -\lambda_0 / (\ln t)^{\psi - 1}. \qquad (8)$$

The asymptotic velocity is determined by the regions of highest mobility if $\lambda_0 > 0$—because then the slower regions can adapt by tilting—but by the low mobility regions if $\lambda_0 < 0$. The purely columnar problem, corresponding to the thermal noise term $\eta(\mathbf{x}, t)$ being absent in (2), has very similar behavior; the only difference lies in the value of the exponent ψ in (6) and (8), which is given by $\psi = 1 + 2/d$ in d dimensions [14].

Figures 2 and 3 show numerical results for the interface width $W(t)$ and the correlation function $G(r,t)$, obtained using the SOS Hamiltonian (4) with $q = 1$ and $q = 2$ at $k_B T = 0.5$ and $F = 1$. Both sets of data are nicely consistent with predictions (6) and (7), if one takes into account the slow approach to asymptopia implied by the logarithmic factor in (6). For example, the last decade of the data for the interface width can be well fitted also by a power law $W \sim t^{\beta}$ with $\beta \approx 0.8$ for $q = 1$ and $\beta \approx 0.72$ for $q = 2$, but a fit of the form (6) is superior in both cases, with an empirical value of $\psi \approx 2$ for $q = 1$ and $\psi \approx 3$ for $q = 2$.

However, a clear difference between the two models shows up in the average interface velocity $v(t)$ (upper inset in Fig. 3). The result for $q = 1$ is consistent with a negative $(\ln t)^{-1}$ correction, corresponding to $\lambda_0 > 0$ in (8); this is confirmed by a direct measurement of the inclination-dependent velocity [18], which shows that v increases with increasing tilt both for individual phase configurations [such as the flat or staggered configurations in Figs. 1(a) and 1(b)] and in the disorder average (lower inset in Fig. 3). By contrast, no correction (8) can be detected for the Gaussian ($q = 2$) model.

I attribute the difference to an extra tilt symmetry specific to the Gaussian model, in any substrate dimension d. Note first that the energy difference entering the transition rates (5) can be written, for $q = 2$, as $\Delta \mathcal{H}_{\pm} = \mp 2(\hat{\nabla}^2 h)_{\mathbf{x}} \mp F + 2d$ where $\hat{\nabla}^2$ denotes the lattice Laplacian. From (3) we have $(\hat{\nabla}^2 h)_{\mathbf{x}} = (\hat{\nabla}^2 n)_{\mathbf{x}} + (\hat{\nabla}^2 \phi)_{\mathbf{x}}$ and hence the dynamics is invariant, for arbitrary fixed phase configuration $\phi_{\mathbf{x}}$, under *integer* tilts, i.e., defining a tilted configuration

$$\tilde{h}_{\mathbf{x}} = h_{\mathbf{x}} + \mathbf{u} \cdot \mathbf{x}; \qquad (9)$$

then if the tilt vector $\mathbf{u}$ has only integer components, the variables $\tilde{n}_{\mathbf{x}} = n_{\mathbf{x}} + \mathbf{u} \cdot \mathbf{x}$ are integers which evolve identically to the $n_{\mathbf{x}}$. It follows in particular that the interface velocity in the one-dimensional Gaussian model is a periodic function of the imposed tilt $u = \langle h_{i+1} - h_i \rangle$, with unit period, for any fixed configuration of phases. One consequence of this symmetry is that the KPZ nonlinearity in the *pure* system is quite weak, and

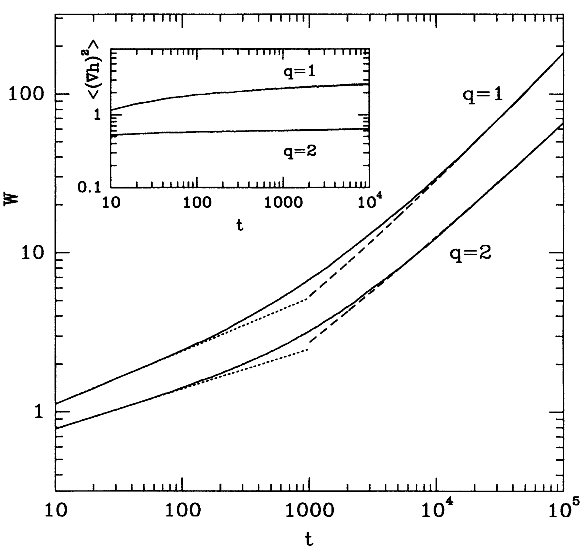

FIG. 2. Simulation results for two versions of the phase-disordered SOS model, obtained from single runs on a system of size $L = 10^5$. Main figure shows interface width as a function of time. Dotted lines are power law fits $W \sim t^{\beta}$, while dashed lines show the asymptotics $W \sim t/(\ln t)^{\psi}$ predicted by (6); the values of the exponents are $\beta = 1/3$, $\psi = 2$ for $q = 1$, and $\beta = 1/4$, $\psi = 3$ for $q = 2$. The inset shows the local gradient $\langle (h_{i+1} - h_i)^2 \rangle$ remains bounded.

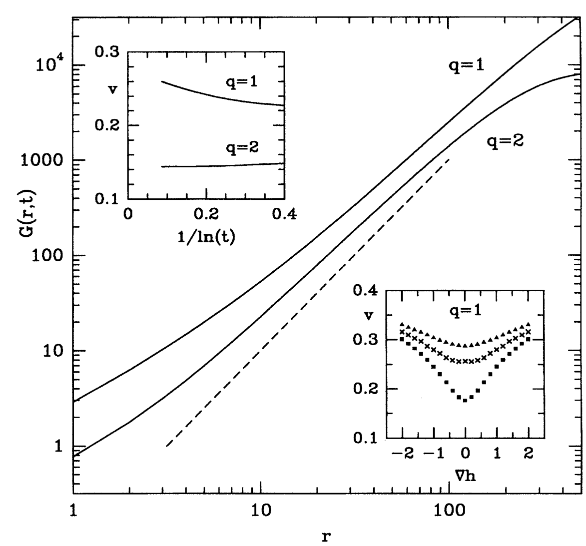

FIG. 3. Main figure shows the height difference correlation function (7) for systems of size $L = 10^5$ at time $t = 10^5$; the dashed line indicates the prediction $G(r) \sim r^2$. Upper inset illustrates the finite time correction to the interface velocity $v = \langle h_i \rangle / t$. Lower inset shows the inclination dependence of the velocity for the $q = 1$ model with a flat (squares) and a staggered (triangles) substrate, as well as in the disorder average (crosses); these data were obtained from simulations of systems of size $L = 1000$.

consequently the early time regime in Fig. 2, during which the disorder is not yet being felt, is governed by the linear theory with $\beta = \frac{1}{4}$ [10].

Upon disorder averaging the discrete tilt symmetry extends to arbitrary *real* tilt vectors. Indeed, defining a new set of variables through $\tilde{n}_{\mathbf{x}} = n_{\mathbf{x}} + [\phi_{\mathbf{x}} + \mathbf{u} \cdot \mathbf{x}]$, $\tilde{\phi}_{\mathbf{x}} = \phi_{\mathbf{x}} + \mathbf{u} \cdot \mathbf{x} - [\phi_{\mathbf{x}} + \mathbf{u} \cdot \mathbf{x}]$, where $[y]$ denotes the integer part of y, we can write $\tilde{h}_{\mathbf{x}} = \tilde{n}_{\mathbf{x}} + \tilde{\phi}_{\mathbf{x}}$, and since $\hat{\nabla}^2 \tilde{h} = \hat{\nabla}^2 h$ from (9), $n_{\mathbf{x}}$ and $\tilde{n}_{\mathbf{x}}$ have the same evolution. Thus the tilted interface evolves like an untilted one with a different set of random phases. Given that all phase configurations have the same statistical weight, we conclude that the disorder-averaged properties of the Gaussian model are tilt independent. This argument carries over [21] to the sine-Gordon equation (1): The tilted field $\tilde{h}(\mathbf{x}, t) = h(\mathbf{x}, t) + \mathbf{u} \cdot \mathbf{x}$ satisfies the same equation as the original field h, if the random phases are replaced by $\tilde{\phi}(\mathbf{x}) = (\phi(\mathbf{x}) + \mathbf{u} \cdot \mathbf{x}) \bmod 1$.

It seems plausible to associate the tilt independence of the disorder averaged problem with the fact that $\lambda_0 = 0$ in (2); if this is correct, then the disordered sine-Gordon equation (1), including its noiseless version used to model CDW's [2,3], would also belong to this new universality class. On the other hand, at least in the context of phase-disordered SOS models, the tilt invariance of the Gaussian model is clearly an anomaly; generically phase-disordered driven interfaces would be expected to behave like the $q = 1$ model (or the restricted SOS model investigated in [15]), with $\lambda_0 \neq 0$, and thus be equivalent to the directed polymer with columnar defects [14]. Nevertheless, it is of interest to gain a better understanding of the $\lambda_0 = 0$ situation, in particular of the relative roles of the random lateral drift $\mathbf{c}$ and the nonlinearity; the *relevance* of these terms is strikingly demonstrated in the simulations by the fact that the local interface gradient $(\nabla h)^2$ remains bounded for the Gaussian model (inset of Fig. 2), instead of diverging with time, as it would in their absence [3]. While the results presented in Figs. 2 and 3 suggest that the behavior for $\lambda_0 = 0$ is rather similar to the generic case $\lambda_0 \neq 0$, this may well be an artifact of one dimension. Preliminary simulations of a lattice model of driven flux lines in the plane [22] (which does not share the tilt symmetry of the Gaussian and sine-Gordon models) show qualitative agreement with the predictions (6) and (7).

The conclusions of this work differ in important respects from those of an earlier investigation [6], where it was claimed that the asymptotics of (1) under strong driving would be governed by the *pure* KPZ equation, since the periodic potential, including the phase disorder, becomes smeared out by the moving interface. I have argued instead that the phase disorder generates a random mobility whose effect persists, in principle, at arbitrarily long times and arbitrarily high temperatures and completely changes the large scale fluctuations. It is clear from the qualitative arguments given for the SOS models that the emergence of the random mobility is related to the *short length-scale cutoff* (i.e., the lattice constant) of the continuum theory, which might be the reason why it was not noted previously.

I have benefited from discussions with D. Cule, M. P. A. Fisher, P. Le Doussal, Y. Shapir, and L. H. Tang. This work was supported in part through NSF Grant No. PHY89-04035.

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