Stability of a Solitary Pulse against Wave Packet Disturbances in an Active Medium

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Solitary pulses in an active medium, like those for the Kuramoto-Sivashinsky equation for thin falling films, are often destroyed by growing and expanding wave packets from localized disturbances. The stability of a solitary pulse is then determined by whether is can escape the expanding wave packets. We show that this pulse-packet interaction is determined by the essential spectrum of the pulse and a pulse stability theory can be derived from a simple extension of the classical convective instability theory for a wave packet on the trivial state.

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Solitary-pulse traveling-wave solutions are localized solutions that propagate at a constant speed c. In a frame traveling at the same speed, they are stationary and their amplitude decays exponentially towards the trivial basic state in both the upstream and downstream directions. Such pulses are frequently observed among deep-water and thin-film waves. The former class gives rise to the classical Korteweg–de Vries (KdV) and Boussinesq solitons [1] while the latter pulses have been constructed from a family of thin-film equations [2], including the Kuramoto-Sivashinsky (KS) equation. Whether these pulses are observable depends fundamentally upon their linear stability. Such pulse stability theories have been formulated for the integrable deep-water waves [3–6]. The spectrum of the solitary pulse consists of a discrete part and a continuous essential part. Because of the integrability, the essential spectrum of an integrable pulse lies on the imaginary axis. Its stability is hence determined entirely by the discrete spectrum. Such instabilities have been observed for forced KdV and Boussinesq solitons in the experiments of Lee [7] and the simulations of Wu [1].

Pulse stability theory for nonintegrable systems like thin-film waves has not been developed at all. The answer seems trivial for an active medium where the trivial state is unstable to spatially periodic disturbances. Part of the pulse essential spectrum hence lies in the right half of the complex plane and the pulse seems unstable. Physically this implies that the primary instability of the trivial state will grow and destroy the pulse. Indeed the positive solitary pulse of KS equation

$$\frac{\partial h}{\partial t} + 4h\frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2} + \frac{\partial^4 h}{\partial x^4} = 0 \qquad (1)$$

with $c_+ = 1.216$, which was constructed in several studies [8,9], has never been observed in various numerical experiments [10]. Because the symmetry of the KS equation to the transformations $t \longrightarrow t$, $x \longrightarrow -x$, and $h \longrightarrow -h$, there is also a negative pulse with speed $c_- = -1.216$ which is an inversion and a reflection of the positive pulse. In Fig. 1, we place two negative pulses behind two positive ones with some initial noise between the pulses. All four solitary pulses are rapidly destroyed.

However, solitary pulses are observed in many active media. In falling films of reasonable thickness [2], for example, pulses dominate most of the downstream wave dynamics. The KS equation describes thin-film waves at very low flow rates and Kapitza [11] has noticed that pulses are not observed for very thin films. However, at slightly higher flow rates, pulses are commonly observed. These thicker film waves can be modeled by the generalized Kuramoto-Sivashinsky (GKS) equation

$$\frac{\partial h}{\partial t} + 4h\frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial x^2} + \delta\frac{\partial^3 h}{\partial x^3} + \frac{\partial^4 h}{\partial x^4} = 0 \qquad (2)$$

or the related averaged equations [9]

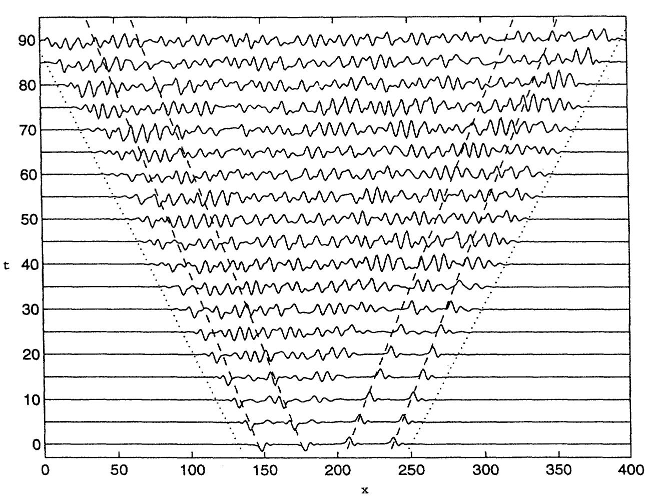

FIG. 1. Numerical experiment for the KS equation with a train of positive and negative pulses. Disturbances are introduced within the train at time zero. The predicted pulse speeds $c_\pm$ and packet envelopes speeds $(x/t)_\pm$ from our theory are also drawn. The h scale can be measured from the t scale with a conversion factor of 0.351.

$$\frac{\partial q}{\partial t} + \frac{6}{5}\frac{\partial}{\partial x}(q^2/h) - \frac{1}{5\delta}\left(h\frac{\partial^3 h}{\partial x^3} + h - q/h^2\right) = 0,$$
$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{3}$$

The speed $c_+(\delta)$ of the GKS solitary pulse family from [9] is shown in Fig. 2. The speeds of the positive and negative pulse families of the averaged equations from [12] are also shown in Fig. 2. in Fig. 3, two positive pulses of the GKS equation are seen to survive the initial noise near the back pulse. In Fig. 4, a positive pulse of the averaged equations is created within a turbulent spot of noise and escapes unscathed. The above numerical experiments suggest that solitary pulses can survive localized disturbances even if their essential spectrum is unstable. (The point spectra will be shown to be stable for both equations.) Localized disturbances tend to grow into wave packets for such systems as seen in Figs. 1,3, and 4. The amplitude and width of these propagating wave packets grow in time as they are driven by the active medium. When the amplitude of the wave packet is small, its width and speed can be determined by the classical convective instability theory for the trivial state [13]. It is then reasonable that there is a connection between the stability of pulses to localized disturbances and the wave packet evolution. We shall establish this connection here and show that the key is whether the pulse can escape the expanding wave packet. This occurs beyond a critical δ for pulses of both the GKS and averaged equations. Hence, as long as the disturbances are small and localized, large-δ pulses in an active medium can survive even though they possess an unstable essential spectrum.

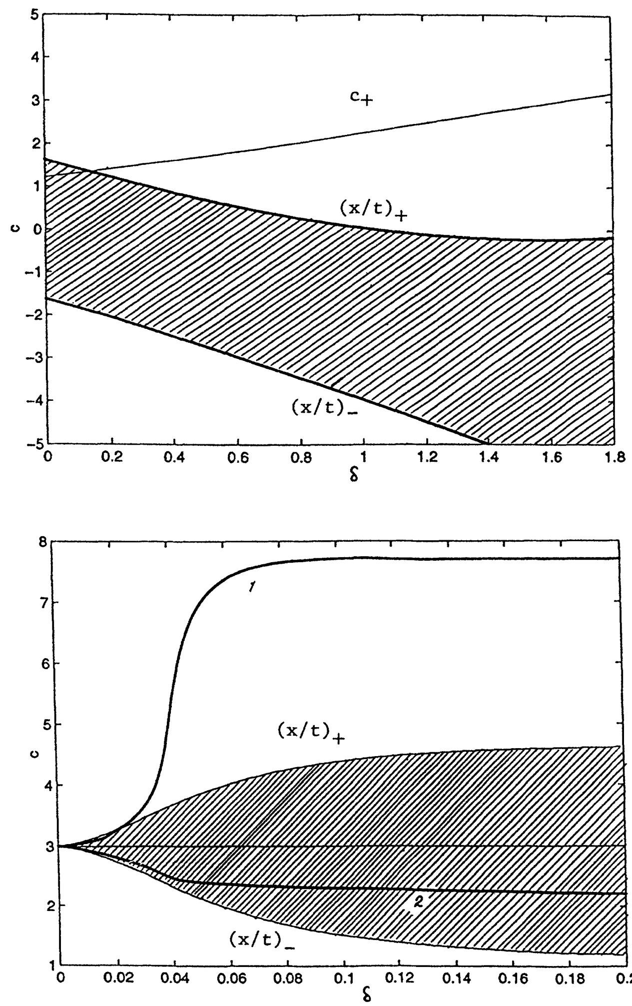

FIG. 2. The speed c_+ of the positive pulse for the GKS equation as a function of δ with the envelope speeds $(x/t)_\pm$. The growth rate γ is positive within the shaded region and the pulse becomes stable for $\delta > 0.17$. The pulse speeds $c_\pm$ of the positive (1) and negative (2) pulses of the averaged equations are shown in the lower figure. The negative pulse is always unstable while the positive pulse becomes stable for $\delta > 0.021$.

Linearizing the equations about the pulse solution $h(y)$ [and $q(y)$] in a frame $y = x - ct$ moving with the pulse speed c, one gets the following equation for the disturbance $H(y,t)$: $\partial H/\partial t - \mathcal{L}H = 0$, where $-\mathcal{L} = \partial^4/\partial y^4 + \delta(\partial^3/\partial y^3) + \partial^2/\partial y^2 + 4(\partial/\partial y)(h\cdot)$ for the GKS and the KS equations and

$$-\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ -\frac{\partial}{\partial y} & c\frac{\partial}{\partial y} \end{pmatrix}$$
$$\mathcal{L}_1 = c\frac{\partial}{\partial y} - \frac{1}{5\delta}\frac{1}{h^2} - \frac{6}{5}\frac{\partial}{\partial y}(2q/h)$$
$$\mathcal{L}_2 = \frac{1}{5\delta}\left[h\frac{\partial^3}{\partial y^3} + \frac{\partial^3 h}{\partial y^3} + 1 + (2q/h^3)\right] + \frac{6}{5} + \frac{\partial}{\partial y}(q/h)^2$$

for the averaged equations where now H is a vector function representing the disturbances in both h and

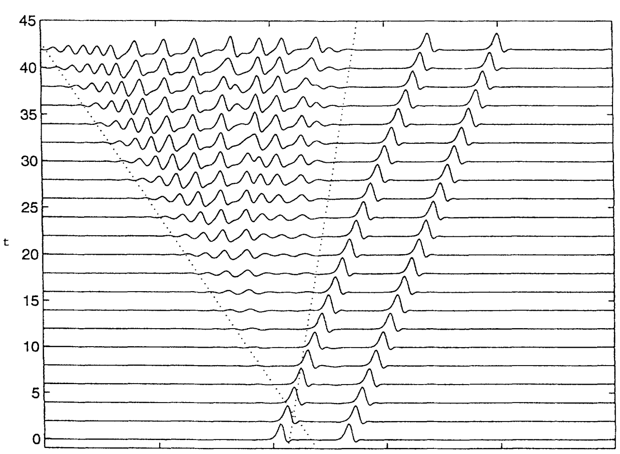

FIG. 3. Numerical simulation of the GKS at $\delta = 0.5$ equation with two positive pulses. The wave packet envelopes $(x/t)_\pm$ from our theory are also drawn. The conversion factor from the t scale to the h scale is 0.692.

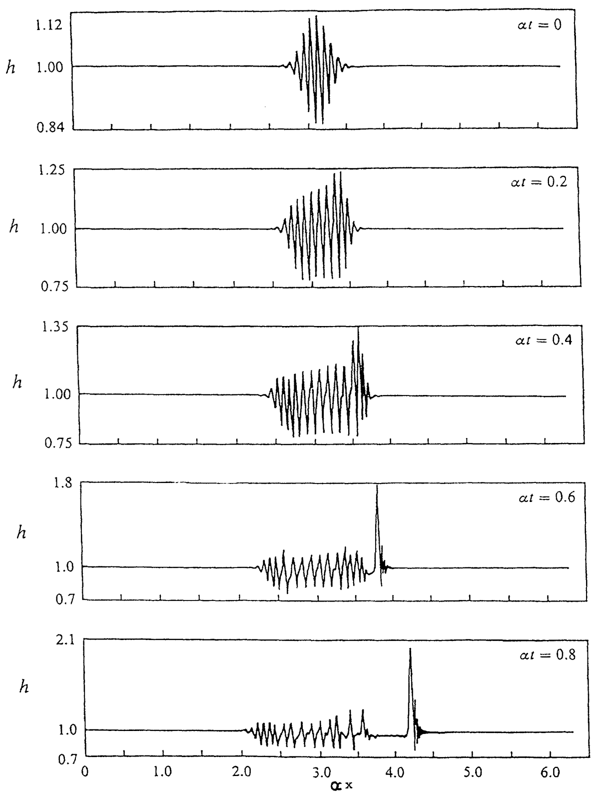

FIG. 4. Numerical simulation of the averaged equation at $\delta = 0.05$. A positive pulse is created within a wave packet but escapes the positive envelope $(x/t)_+$. The parameter α is 0.008 97.

q. The corresponding initial and boundary conditions are $H(y, t = 0) = H_0(y)$ with $H_0(\pm\infty) \to 0$ and $H(y \to \pm\infty, t) \to 0$. The localized nature of H_0 and H allows us to construct proper transforms. It also disallows global disturbances which are known to be unstable.

Carrying out a Laplace transform with respect to time $\hat{H}(y,p) = \int_0^\infty H(y,t)e^{-pt}\,dt$ one gets a boundary value problem for the transform $\hat{H}$, $(p - \mathcal{L})\hat{H} = H_0(y)$ with $\hat{H}(y \to \pm\infty) \to 0$. We solve this boundary value problem by using Green's function $\hat{H}(y,p) = \int_{-\infty}^{\infty} G(y,\xi,p)H_0(\xi)\,d\xi$ where $(p - \mathcal{L})G(y,\xi,p) = \delta(y - \xi)$. Consequently, the full solution $H(y,t)$ is provided by the transform integral

$$H(y,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\int_{-\infty}^{\infty} G(y,\xi,p)H_0(\xi)e^{pt}\,d\xi\,dp\,, \tag{4}$$

where the line $p = a$ lies to the right of the spectrum of $\mathcal{L}\psi = \lambda\psi$ due to causality considerations [$H(y,t) = 0$ for $t < 0$].

The spectrum of $\mathcal{L}$ has two parts. The discrete spectrum $\{\lambda_k\}$ corresponds to eigenfunctions that decay to zero at the infinities, $\psi_k(y \to \pm\infty) \to 0$ and the continuous essential spectrum λ whose eigenfunctions are merely bounded at the infinities [6]. Because of the translational invariance of the equation to $x \to x + c$, there is always a simple zero within the discrete spectrum corresponding to the eigenfunction $(\partial h/\partial y)$ or $(\partial h/\partial y, \partial q/\partial y)$ [14–16]. This simple zero cannot destabilize the pulse and one must consider the continuous spectrum if all other discrete eigenvalues are stable. Excitation of the continuous spectrum can occur when two pulses coalesce irreversibly to form a single perturbed pulse [16]. The continuous spectrum can also excite "radiation modes" which generate additional pulses [15] (see Figs. 1 and 3). There can hence be complex interaction between the unstable discrete and continuous spectra. The continuous spectrum is defined by the linear dispersion relationship in the moving frame. For the GKS equation, for example, $\lambda = i(\alpha c + \delta\alpha^3) + \alpha^2 - \alpha^4$ and is parametrized by the wave number α. For the averaged equations with a matrix operator $\mathcal{L}$, the dispersion relationship is obtained by taking the determinant of $\lambda I - \mathcal{L}$ where $\partial/\partial y$ is replaced by $i\alpha$ [17]. The eigenfunctions corresponding to the continuous spectrum $\psi(y,\lambda)$ approach a sine function at $\pm\infty$ with spatial wave number α and amplitudes unity and r (the transmission coefficient) at $-\infty$ and $+\infty$, respectively. It is more convenient to parameterize the continuous spectrum $\lambda(\alpha)$ by the wave number α on the real line and decompose the corresponding eigenfunction into a phase and amplitude part $\psi(y,\lambda) = K(y,\alpha)e^{i\alpha y}$. We have numerically constructed the discrete and continuous spectra for the KS operator. Other than the simple zero within the discrete spectrum corresponding to the translation mode $\partial h/\partial y$, we found only one other discrete eigenvalue, and it is stable. In fact the discrete spectra of all the positive and negative pulses of the GKS and averaged equations are found to lie in the left half plane [16]. The continuous eigenfunctions $\psi(y,\lambda)$ are difficult to construct, but our theory does not require their explicit construction.

Knowing the spectrum of $\mathcal{L}$ specifies the poles of Green's function in Eq. (4) and applying the residue theorem for the Laplace integral yields $H(y,t) = \sum_k A_k\psi_k(y)e^{\lambda_k t} + \int A(\lambda)\psi(y,\lambda)e^{\lambda t}\,d\lambda$ where A_k and $A(\lambda)$ are the expansion coefficients of the discrete and continuous eigenfunctions for the initial condition $H_0(y)$. Their exact values are not important as long as $H_0(y)$ is sufficiently rich such that the pertinent $A(\lambda)$ is not zero. Note that while $\psi(y,\lambda)$ is not localized $H(y,t)$ is necessary as required for functions which can be Fourier transformed. In essence, the Fourier transform is replaced by an expansion in terms of the continuous eigenfunctions. Since the discrete spectra are stable, the large-time behavior of $H(y,t)$ is determined by $H(y,t) \to \lim_{t\to\infty}\int_{-\infty}^{\infty} A(\alpha)K(y,\alpha)e^{i[\alpha y/t + \alpha c - \omega]t}\,d\alpha$ where we have replaced λ by α, separated the phase part of the eigenfunction, and used the substitution $\lambda = i\alpha c - i\omega$, where ω is the complex wave frequency for the trivial state in the stationary frame. The large-time behavior of the Fourier integral can be determined by the stationary phase (or steepest descent) technique. For a given ray (y/t constant), the behavior of $H(y,t)$ is determined by the saddle stationary point defined by $(\partial/\partial\alpha)[\alpha c - \omega] = c - \partial\omega/\partial\alpha = -(y/t)$. Since we are in the moving frame y where the pulse is located at $y/t \rightarrow 0$, the pertinent behavior of $H(y,t)$ is at $y/t = 0$ dominated by the saddle point α_* where $(\partial\omega/\partial\alpha)(\alpha_*) = c$ and instability of the pulse occurs if $\mathrm{Im}\{\alpha_* c - \omega(\alpha_*)\} > 0$.

These conditions can also be derived from the perspective of the wave packet in the fixed frame x. According to the classical convective stability theory [13], a localized disturbance $\delta(x)\delta(t)$ on the trivial basic state will grow into a wave packet in the x frame and the wave number selected along the (x/t) rays is defined by $(\partial\omega/\partial\alpha)(\alpha_*) = (x/t)$ and whether the disturbance will grow or decay along this ray is determined by the sign of $\mathrm{Im}\{\alpha_* x/t - \omega(\alpha_*)\}$. This is derived from the appropriate Fourier integral with a similar stationary phase argument. Hence the two boundaries of the wave packet are defined by the rays $(x/t)_\pm$ where $(\partial\omega/\partial\alpha)(\alpha_\pm) = (x/t)_\pm$ and $\mathrm{Im}\{(\alpha x/t)_\pm - \omega(\alpha_\pm)\} = 0$.

Comparing these conditions to the pulse stability condition at the saddle point α_*, it is clear that if the solitary wave speed c is such that $(x/t)_- < c < (x/t)_+$ the solitary pulse is unstable. Conversely if c is larger than $(x/t)_+$ or smaller than $(x/t)_-$, the pulse is stable. This then links the stability of a pulse to a wave packet disturbance to the classical convective instability theory for a localized disturbance on the trivial state. While the former involves expansion with the continuous eigenfunctions and the latter Fourier expansion, the pertinent saddle points are identical for both cases.

For the GKS, the complex wave number $\alpha_\pm$ are defined by $4\alpha^3 - 3i\delta\alpha^2 - 2\alpha - i(x/t) = 0$ for any given x/t. This complex polynomial has three roots for $\delta \neq 0$ and two roots for the KS limit at $\delta = 0$. Only one of the roots corresponds to the true saddle point and we utilize the classical complex pinch-point analysis method [13] to determine which is the true saddle point. The growth rate along this particular (x/t) ray $\gamma = \mathrm{Im}\{\alpha x/t - \omega(\alpha)\}$ is then evaluated at the proper root. The envelope rays $(x/t)_\pm$ are then defined by $\gamma(x/t) = 0$. For the symmetric KS case these values are ± 1.622 and they bound $c_\pm = \pm 1.216$ of both the positive and negative pulses of the KS equation. In Fig. 1, both $c_\pm$ and $(x/t)_\pm$ are shown and it is quite evident that the pulses cannot escape the expanding wave packet. For the GKS equation at $\delta = 0.5$, the envelope rays are $(x/t)_+ = 0.704$ and $(x/t)_- = -2.728$ such that the positive pulse with $c_+ = 1.709$ is now stable while the negative pulse, which is not shown in Fig. 2, remains unstable. In Fig. 3, the estimates $(x/t)_\pm$ are seen to envelop the growing wave packet while the positive pulses safely escape their grip as predicted. A sequence of such calculations provides the $(x/t)_\pm$ curves in Fig. 2 and they indicate that the positive pulse becomes stable to wave packet disturbances at $\delta = 0.17$. The pinch-point calculations and the specification of the envelope speeds $(x/t)_\pm$ for the averaged equations are more complicated and they are reported elsewhere [17]. The final result shown in Fig. 2 indicates the negative pulse is always unstable while the positive pulse becomes stable at $\delta = 0.021$, confirming the numerical simulation result of Fig. 4 for $\delta = 0.05$ where it is apparent that $c_+ > (x/t)_+$.

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