Chaos in Andreev Billiards

Ioan Kosztin,[1] Dmitrii L. Maslov,[1,2,4] and Paul M. Goldbart[1-3]
[1]*Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801*
[2]*Materials Research Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801*
[3]*Beckman Institute, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801*
[4]*Institute for Microelectronics Technology, Academy of Sciences of Russia, Chernogolovka, 142432 Russia*
(Received 24 April 1995)

A new type of classical billiard—the Andreev billiard—is investigated using the tangent map technique. Andreev billiards consist of a normal region surrounded by a superconducting region. In contrast with previously studied billiards, Andreev billiards are integrable in zero magnetic field, *regardless of their shape*. A magnetic field renders chaotic motion in a generically shaped billiard, which is demonstrated for the Bunimovich stadium by examination of both Poincaré sections and Lyapunov exponents. The issue of the feasibility of certain experimental realizations is addressed.

PACS numbers: 05.45.+b, 74.80.–g, 74.40.+k

In the development of the understanding of chaos, a prominent role has been played by the class of classical mechanical systems known as billiards [1]. In such systems, particles are confined by a steplike, single-particle potential to a region of space within which they propagate ballistically. Unless the shape of the billiard is highly regular (e.g., circular), in which case the system is integrable, the motion of a particle is chaotic. Unpredictability, the hallmark of chaotic motion, can be diagnosed qualitatively by the morphologies of Poincaré sections, and more quantitatively by the corresponding Lyapunov exponents, the positivity of which signals the exponential sensitivity of trajectories to initial conditions.

A common feature of all versions of billiards studied to date [1] is that reflection at boundaries is specular; i.e., only the component of the velocity *normal* to the boundary is inverted. We refer to such versions as conventional billiards (CB's); see Fig. 1(a), left. The purpose of this Letter is to explore the issue of classical chaos in a novel class of billiards, which have the property that scattering at boundaries is retroreflective, i.e., *all* components of the velocity are inverted, as is depicted in Fig. 1(a), right. We refer to such billiards as Andreev billiards (AB's). Although we are unaware of examples of such a reflection mechanism in the realm of classical physics, a well-known example exists in condensed matter physics: Andreev reflection of electronic quasiparticles (having energies in the superconducting gap Δ) from the normal-to-superconductor interface [2,3]. Thus, we envisage AB's as normal (N) domains surrounded by superconductor (S). It is adequate to regard the motion of electronic quasiparticles as semiclassical, provided that the billiard size is much larger than their typical de Broglie wavelength.

The change from specular to Andreev reflection has a striking consequence in the context of chaos: Whereas typical motion in a generically shaped CB is chaotic, motion in AB's is integrable, regardless of the shape of the billiard. This integrability becomes evident from the observation [cf. Fig. 1(a)] that all motion occurs along chordal trajectories connecting only two points on the boundary: what system could be less ergodic?

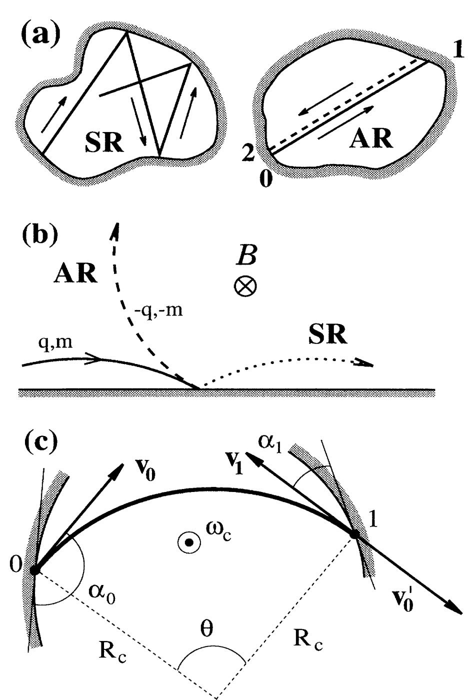

FIG. 1. (a) Typical trajectories for specular (SR) and Andreev (AR) reflection at $B = 0$. The CB (left) is formed by a single-particle potential, the AB (right) by a pair potential. (b) AR and SR in a magnetic field. For AR, each cyclotron orbit is necessarily tangential to the previous one, in contrast with SR. (c) Geometry of the tangent map. A particle starts from 0 with velocity v_0, follows the cyclotron trajectory across the billiard, arriving at 1 with velocity v_0'. Owing to the nature of AR, the velocity v_1 after the reflection at 1 is $-v_0'$.

The presence of a magnetic field **B** substantially alters the situation [see Fig. 1(b)], giving rise to a Lorentz force that curves the trajectories $\mathbf{r}(t)$ of quasiparticles according to the equation of motion $\ddot{\mathbf{r}} = (q/m)\dot{\mathbf{r}} \times \mathbf{B}$. Now, Andreev reflection inverts the quasiparticle charge q, mass m, and velocity $\dot{\mathbf{r}}$, so that in the vicinity of the reflection point the acceleration of the outgoing hole is opposite to that of the incoming electron. Therefore, in a magnetic field the hole trajectory [dashed line in Fig. 1(b)] no longer retraces the electron trajectory (full line) [4], and vice versa, thus allowing the motion to explore the billiard. This raises the possibility of chaotic motion, a possibility that we explore in this paper. Our primary conclusion is that although in zero **B** AB's are integrable regardless of their shape, integrability is destroyed by the application of magnetic field for all but highly regular shapes.

The study of AB's may provide an interesting link between two rapidly developing fields: mesoscopic chaos [5] and mesoscopic superconductivity [6]. The prominent virtue of AB's, viz., that they are integrable in zero **B** regardless of shape and are rendered chaotic by nonzero **B**, makes them attractive from the experimental point of view. By comparison, the integrability of nanoscale CB's (e.g., in two-dimensional electron gas (2DEG) heterostructures [5]) is immensely fragile, being readily destroyed by unintentional shape deformations or surface roughness. Therefore, to obtain nontrivial chaos in CB's, i.e., chaos caused by intentional choice of shape, one must use state-of-the-art nanofabrication technology. In contrast, an AB prepared without any special attention to shape will be integrable, nonintegrability of varying degree being achieved by adjusting an external parameter, viz., **B** [7].

To explore qualitatively the implications of a magnetic field for the integrability of AB's, we focus on a planar two-dimensional AB in a magnetic field perpendicular to the plane. We take the dynamics to be classical cyclotron motion with radius R_c ($\equiv mv/qB$) of a particle inside the billiard, supplemented by Andreev reflection at the boundary. The case of CB's in a magnetic field has been studied extensively [8,9], and the tangent map [9,10] has proven to be a convenient approach. A tangent map is a variant of a Poincaré map, in which the state of the system is monitored only at collisions with the boundary, the remainder of the motion being obtained by a simple geometry. Thus, the problem reduces to one of following the sequence of reflection points generated by the particle as it explores the billiard.

A priori, our phase space is four-dimensional: two components of the position in the plane and two conjugate momenta. Energy conservation constrains the magnitude of the momentum, leaving one freedom, which (as we are using the tangent map) we take to be the angle α between the velocity **v** and the tangent to the boundary [see Fig. 1(c)]. In addition, the fact that reflections take place on the boundary leaves one further freedom, which we take to be the arclength s along the boundary. Then the (continuous time) dynamics is replaced by the (discrete) map: $\{s, \alpha\} \rightarrow \{s'(s, \alpha), \alpha'(s, \alpha)\}$, embodying cyclotron motion followed by Andreev reflection. For the sake of convenience, we monitor only reflections of quasiparticles of the same (say, electron) type. [Thus, e.g., for the sequence of reflections $0 \rightarrow 1 \rightarrow 2$ shown in Fig. 1(a), only reflections 0 and 2 are used to construct Poincaré sections.]

First, consider the case of an AB of arbitrary shape at $B = 0$. In this case, all trajectories, such as that depicted in the right billiard of Fig. 1(a), are trivially periodic, and the Poincaré section for a given trajectory reduces to a single point, completely determined by the initial conditions. We analyze the case of $B \neq 0$ for the example of an AB in the shape of a Bunimovich stadium [11], as shown in the top row of Fig. 2. Figure 2 shows Poincaré sections (bottom row) for a selection of initial conditions, along with typical trajectories (top row). For convenience, we introduce the dimensionless magnetic field $\beta \equiv R/R_c \propto B$ and the tangential momentum $p \equiv \cos\alpha$. For the case of a weak field ($\beta = 0.02$, left column), $\{s, p\}$ space is apparently foliated by well-defined curves, each curve corresponding to a particular initial condition. Although it appears that the motion is integrable, when viewed at a finer scale one sees the breakdown of foliation, as shown in the inset. Thus, the motion is in fact weakly chaotic, as we have also confirmed by examining the corresponding Lyapunov exponent. In intermediate fields (for which the cyclotron radius is comparable to the billiard size), the Poincaré sections (except those in a central region) appear to fill a two-dimensional area of the $\{s, p\}$ plane with disconnected points ("dust"), as shown for the case $\beta = 0.33$ in the middle column. Such behavior is commonly taken as an indication of chaos [9,10], and thus our Poincaré sections suggest that indeed AB's are rendered chaotic by the application of magnetic field. Notice the scarlike feature running across the $\{s, p\}$ plane: it is a remnant of the foliation that dominates in weaker fields. A single, additional, quasi-one-dimensional Poincaré section, corresponding to an initial condition deliberately chosen in the scar, is also shown. In strong fields, when the cyclotron radius is much smaller than the billiard size, particles move along skipping trajectories. On the scale of a typical skip, there is little distinction between motion over straight and semicircular segments of the boundary and, therefore, the motion is less sensitive to the billiard shape. As a result, the Poincaré section exhibits a (partial) reentrance of integrability, i.e., the "dust" that arises in intermediate fields is reorganized into some structures, as is seen in the bottom row (for $\beta = 10$). This structure bears a certain resemblance to that found in weak fields, thus indicating a trend toward less chaotic behavior. As with CB's [9], chaos is

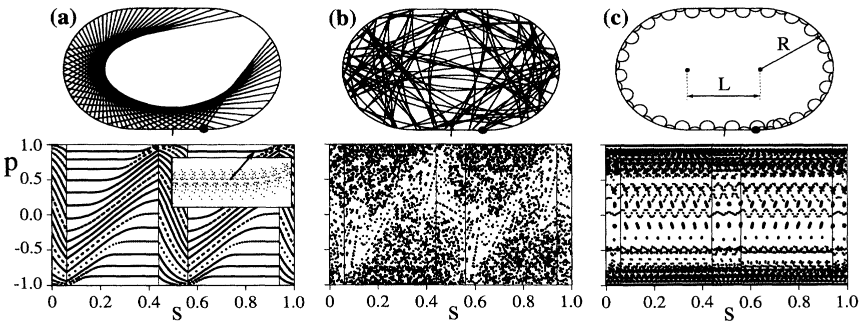

FIG. 2. Top row: a typical trajectory for a Bunimovich-stadium–shaped AB ($L = R$) for three values of the magnetic field: (a) $\beta = 0.02$; (b) $\beta = 0.33$; (c) $\beta = 10$. Bottom row: Poincaré sections in these fields constructed by following the first 1000 bounces for the trajectories starting with $\alpha_0 = 10°, 20°, \ldots, 170°$ [cf. Fig. 1(c)] from random points on the perimeter of the billiard. Thin vertical lines on Poincaré sections separate regions corresponding to straight (wider) and semicircular (narrower) segments of the billiard boundary. In the Poincaré sections for the weak field, flat segments result from almost-chordal motion across a single semicircle. (*Only* flat regions would occur for a circular billiard.) Similarly, curved regions result from trajectories connecting any two of the four distinct segments of the boundary. Inset: segment (indicated by arrow) of the foliation, magnified ($x\times \sim 10$; $y\times \sim 50$). Ticks on the stadium boundaries mark the points $s = 0$; filled circles indicate the start of trajectories s_0. We choose units in which the billiard perimeter is unity.

most pronounced for intermediate fields, becoming less pronounced in both the weaker and stronger field regimes.

To provide quantitative support for the suggestion of chaos inferred from the inspection of Poincaré sections, we now turn to the computation of Lyapunov exponents, which characterize the rate of exponential divergence of trajectories having initial conditions nearby in phase space. This is accomplished by investigating the stability of the tangent map via the adaptation to AB's of the method of Refs. [9,10]. Consider the situation depicted in Fig. 1(c). From the kinematics of circular motion we have $\boldsymbol{v}'_0 - \boldsymbol{\omega}_c \times \boldsymbol{r}_1 = \boldsymbol{v}_0 - \boldsymbol{\omega}_c \times \boldsymbol{r}_0$, where $\boldsymbol{\omega}_c = q\mathbf{B}/m$ and $\boldsymbol{r}_{0,1}$ are the radius vectors of the reflection points; other notations are defined in Fig. 1(c). The tangent map is derived by varying this equation with respect to s and p, and relating the deviations of two nearby trajectories $\delta\mathbf{q}$ ($\equiv\{\delta s, \delta p\}$) before ($\delta\mathbf{q}_0$) and after ($\delta\mathbf{q}_1$) reflection from point 1: $\delta\mathbf{q}_1 = \mathcal{T}_{1,0}\delta\mathbf{q}_0$. After some straightforward algebra, we find that $\mathcal{T}_{1,0}$ is given by

$$\begin{pmatrix} \dfrac{R_c\sigma(\chi)}{\rho_0\sigma(\alpha_1)} - \dfrac{\sigma(\alpha_0-\chi)}{\sigma(\alpha_1)} & \dfrac{-R_c\sigma(\chi)}{\sigma(\alpha_0)\sigma(\alpha_1)} \\ \dfrac{R_c\sigma(\chi)}{\rho_0\rho_1}\dfrac{\sigma(\alpha_0-\alpha_1-\chi)}{R_c} - \dfrac{\sigma(\alpha_0-\chi)}{\rho_1} - \dfrac{\sigma(\alpha_1+\chi)}{\rho_0} & \dfrac{\sigma(\alpha_1+\chi)}{\sigma(\alpha_0)} - \dfrac{R_c\sigma(\chi)}{\rho_1\sigma(\alpha_0)} \end{pmatrix},$$

where $\rho_{0,1}$ are the radii of curvature of the billiard boundary at the reflection points 0 and 1, $\sigma(\phi) \equiv \sin\phi$, and χ is defined in Fig. 1(c) [12]. After N bounces, the separation $\delta\mathbf{q}_N$ is determined by the matrix $\mathcal{T}_{N,0} = \prod_{j=1}^{N}\mathcal{T}_{j,j-1}$. The largest Lyapunov exponent associated with a given trajectory is calculated as $\lambda = \lim_{N\to\infty}\lambda_N$, where

$$\lambda_N = N^{-1}\ln\left| |\mathrm{Tr}\,\mathcal{T}_{N,0}/2| + \sqrt{(\mathrm{Tr}\,\mathcal{T}_{N,0}/2)^2 - 1}\right|. \tag{1}$$

We have calculated the Lyapunov exponents for a wide selection of initial conditions $\{s_0, p_0\}$ and values of B. A typical sequence λ_N is shown in Fig. 3. The convergence of λ_N to a nonzero value as $N \to \infty$ provides evidence for the exponential divergence of nearby trajectories, i.e., chaos.

We now turn to the issue of possible experimental realizations of AB's. In one possible scheme, an AB is formed by surrounding a 2DEG with a superconducting contact [13]. The chaotic nature of the motion can be diagnosed either by passing normal current through the structure and measuring the conductance, as with nanoscale CB's [5], or by studying, e.g., via scanning tunneling microscopy (STM), correlations in real and energy space, which provide signatures of classical chaos at the quantum level [14]. The primary demand on the experimental realization of all billiards, including AB's, is that the motion of the electrons inside the billiard be ballistic. Our analysis of Poincaré sections and corresponding Lyapunov exponents shows that when $R_c \sim L \sim R$, chaos is established after a few bounces, and thus it will not be masked by impurity scattering provided that $L, R \ll \ell_e$, where ℓ_e is the elastic mean free path of the N region. On the other hand, B should not exceed the (lower) critical field of the superconductor B_c and, hence, $R_c > R_c^{\min} = p_F/eB_c$, where p_F refers to the N region. Thus, it is sufficient to have

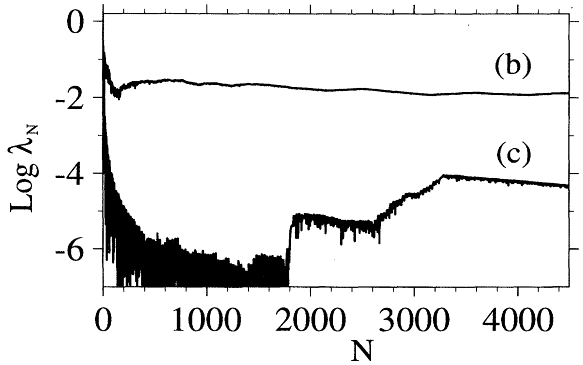

FIG. 3. Lyapunov functions λ_N [logarithmic scale; see Eq. (1)] vs number of bounces N for (b) $\beta = 0.33$; (c) $\beta = 10$. The associated trajectories are shown at the top of Fig. 2. For $\beta = 0.02$ (not shown) we find $\lambda \approx 0.002$.

$\ell_e \gtrsim R_c^{\min}$. Taking parameters for the Nb/InAs structure studied recently (density of electrons $n_e = 9 \times 10^{11}\,\mathrm{cm}^{-2}$, Ref. [13]; $B_c \approx 2000\,\mathrm{G}$), we obtain $\ell_e \gtrsim 0.6\,\mu\mathrm{m}$, which is accessible via current nanofabrication technologies. (In fact, $\ell_e \approx 0.75\,\mu\mathrm{m}$ in Ref. [13].) A drawback of the scheme described above is that the superconductor and the 2DEG are metallurgically distinct, and thus the probability for normal scattering at the interface is nonzero, at the expense of the Andreev reflection, which results in billiards having a mixed AB-CB character. This drawback can be eliminated by employing the proximity effect. The AB is formed in a region of a superconductor where the superconductivity has been suppressed, either due to the vicinity of a normal metal island (see Ref. [15]), or (with a type I superconductor) by the application of a magnetic field, which creates domains of N phase. As such a scheme eliminates metallurgical boundaries between N and S, reflection is of purely the Andreev type.

An interesting direction of further research would be to explore the quantum mechanics of ABs. At least three directions are immediately apparent: (i) spectral geometry and the Weyl-Kac problem; (ii) energy-level statistics, random matrix approaches, and universality; and (iii) spatial structure of quasiparticle wave functions.

We thank D.K. Campbell, S.-J. Chang, C. Eberlein, E.A. Jackson, J.P. Sethna, M. Tabor, and D.J. Van Harlingen for useful discussions. This work was supported in part by NSF via Grants DMR91-20000 (through STCS, IK), DMR-20538 (through MRL, DLM), and DMR94-24511 (PMG).

[1] See, e.g., Ref. [10], and references therein.
[2] A.F. Andreev, Zh. Eksp. Teor. Fiz. **46**, 1823 (1964); **49**, 655 (1965) [Sov. Phys. JETP **19**, 1228 (1964); **49**, 455 (1966)].
[3] Andreev reflection is the process in which an electronlike quasiparticle, say, moving in N impinges on an N/S interface and is converted into a holelike quasiparticle, which retraces the trajectory of the incoming electron (a Cooper pair being injected into S). Retroreflection is not quite perfect, due to both electron-hole interconversion and (in a magnetic field) the screening supercurrent that circumnavigates the billiard. However, in both cases the violation of momentum conservation is small (of order $p_F\Delta/E_F \ll p_F$, where p_F is the Fermi momentum and E_F is the Fermi energy) and should be negligible.
[4] See, e.g., V.F. Gantmakher and Y.B. Levinson, *Carrier Scattering in Metals and Semiconductors* (North-Holland, Amsterdam, 1990), p. 261.
[5] For recent experiments and references to earlier work, see C.M. Marcus *et al.,* Phys. Rev. Lett. **69**, 506 (1992); Phys. Rev. B **48**, 2460 (1993).
[6] See, e.g., *Mesoscopic Superconductivity,* in Proceedings of NATO Adv. Res. Workshop, edited by F.W.J. Hekking, G. Schön, and D.V. Averin [Physica (Amsterdam) **203B**, Nos. 3 and 4 (1994)].
[7] In an AB surrounded entirely by superconductor, flux quantization would restrict the allowed values of B. This restriction can be avoided by introducing a radial insulating strip into the superconductor.
[8] K. Nakamura and H. Thomas, Phys. Rev. Lett. **61**, 247 (1988).
[9] O. Meplan, F. Brut, and C. Gignoux, J. Phys. A **26**, 237 (1992).
[10] M.V. Berry, Eur. J. Phys. **2**, 91 (1981).
[11] L.A. Bunimovich, Funct. Anal. Appl. **8**, 254 (1974); Commun. Math. Phys. **65**, 259 (1979).
[12] One can check that det $\mathcal{T}_{1,0} = -1$; the tangent map is area preserving. Moreover, $\lim_{R_c\to\infty} \mathcal{T}_{2,1}\mathcal{T}_{1,0} = \mathbb{1}$, consistent with chordal motion in zero B.
[13] A related structure (Nb/InAs) has recently been fabricated: A. Dimoulas *et al.,* Phys. Rev. Lett. **74**, 602 (1995).
[14] See, e.g., M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
[15] For an STM technique particularly suitable for studying N/S interfaces, see S.H. Tessmer, D.J. Van Harlingen, and J.W. Lyding, Phys. Rev. Lett. **70**, 3135 (1993).