Coulomb Phase of $N = 2$ Supersymmetric QCD

Philip C. Argyres,[1] M. Ronen Plesser,[1] and Alfred D. Shapere[2]
[1]*School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540*
[2]*Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506*
(Received 19 May 1995)

We present an explicit nonperturbative solution of $\mathcal{N} = 2$ supersymmetric SU(N) gauge theory with $N_f \le 2N$ flavors generalizing results of Seiberg and Witten for $N = 2$.

PACS numbers: 11.30.Pb, 11.15.Tk, 12.38.Au

Seiberg and Witten [1,2] have given an exact nonperturbative solution for the low-energy dynamics of $\mathcal{N} = 2$ supersymmetric SU(2) QCD with $N_f \le 4$ flavors. Their solution exhibits confinement and chiral symmetry breaking driven by the condensation of magnetic monopoles with global symmetry charges. For $N_f = 4$ it also exhibits exact scale invariance and strong-weak coupling duality.

In this Letter we construct the generalization of their work to SU(N) with $N_f = 2N$. Our solution, being exact, makes possible the study of strong-coupling phenomena in these theories. It also exhibits exact scale invariance and strong-weak coupling duality. All asymptotically free $\mathcal{N} = 2$ SU(N) theories with $N_f < 2N$ are obtained as appropriate limits of our solution in parameter space.

$\mathcal{N} = 2$ QCD is described in terms of $\mathcal{N} = 1$ superfields by a chiral field strength multiplet W^a_b and a chiral multiplet Φ^a_b in both the adjoint of the gauge group and chiral multiplets Q^i_a in the $\mathbf{N}$ and $\tilde{Q}^a_i$ in the $\overline{\mathbf{N}}$ representations of the gauge group. Flavor and color indices are, respectively, $i, j, k = 1, \ldots, N_f$ and $a, b, c = 1, \ldots, N$.

The Lagrangian contains $\mathcal{N} = 1$ gauge-invariant kinetic terms for the fields with gauge coupling constant $\tau = \theta/\pi + i(8\pi/g^2)$, and the superpotential $\mathcal{W} = \sqrt{2}\tilde{Q}^a_i \Phi^b_a Q^i_b + \sqrt{2} M^i_j \tilde{Q}^a_i Q^j_a$. The quark mass matrix satisfies $[M, M^\dagger] = 0$, implying that it can be diagonalized by a flavor rotation to $M = \mathrm{diag}(m_1, \ldots, m_{N_f})$. Classically and with $M = 0$, the global symmetries are a $\mathrm{U}(N_f)$ symmetry and a $\mathrm{U}(1)_R \times \mathrm{SU}(2)_R$ chiral R symmetry. The trace of the mass matrix M is a flavor singlet, while the rest transforms in an adjoint flavor representation. We denote the flavor-singlet and flavor-adjoint masses by, respectively, $\mu \equiv (1/N_f)\sum_{j=1}^{N_f} m_j$ and $\mu_j \equiv m_j - \mu$.

The theory has a rich vacuum structure. In this Letter we study its Coulomb phase, where the vacuum expectation values (VEVs) of the lowest components of the chiral superfields satisfy $q^i_a = \tilde{q}^a_i = 0$ and $[\phi, \phi^\dagger] = 0$. This implies that ϕ can be diagonalized by a color rotation to a complex traceless matrix $\langle\phi\rangle = \mathrm{diag}(\phi_1, \ldots, \phi_N)$. As gauge-invariant coordinates on the Coulomb phase moduli space, we take the elementary symmetric polynomials s_ℓ defined by $\det(x - \langle\phi\rangle) = \prod_a (x - \phi_a) = \sum_{\ell=0}^N s_\ell x^{N-\ell}$. Note that $s_0 = 1$ and $s_1 = 0$ by the tracelessness of ϕ.

$\langle\phi\rangle$ generically breaks the gauge symmetry SU(N) to $\mathrm{U}(1)^{N-1}$ and gives all the quarks masses, so the low-energy effective theory is an $\mathcal{N} = 2$ supersymmetric $\mathrm{U}(1)^{N-1}$ Abelian gauge theory. Classically, if some of the ϕ_a's are equal, the unbroken gauge group will include non-Abelian factors. Also, when $\phi_a + m_i = 0$ the quark Q^i_a is massless.

If we assume that $\mathcal{N} = 2$ supersymmetry is not dynamically broken, then the Coulomb vacua are not lifted by quantum effects. At a generic point, the low-energy effective Lagrangian can be written in terms of the $\mathcal{N} = 2$ U(1) gauge multiplets (A_μ, W_μ), where $\mu, \nu = 1, \ldots, N-1$ and label quantities associated to each of the U(1) factors. We denote the scalar component of A_μ by a_μ, which we will also take to stand for its VEV.

The $\mathcal{N} = 2$ effective Lagrangian is determined by an analytic prepotential $\mathcal{F}(A_\mu)$ and takes the form

$$\mathcal{L}_{\mathrm{eff}} = \mathrm{Im}\frac{1}{4\pi}\left[\int d^4\theta\, A^\mu_D \overline{A}_\mu + \frac{1}{2}\int d^2\theta\, \tau^{\mu\nu} W_\mu W_\nu\right], \tag{1}$$

where the dual chiral fields and the effective couplings are given by $A^\mu_D \equiv \partial\mathcal{F}/\partial A_\mu$ and $\tau^{\mu\nu} \equiv \partial^2\mathcal{F}/\partial A_\mu \partial A_\nu$. Typically, this effective action is good for energies up to the mass of the lightest massive particle. There are special submanifolds of moduli space where extra states become massless. As we approach these submanifolds the range of validity of (1) shrinks to zero; on these singular submanifolds the effective Lagrangian must be replaced with one that includes the new massless degrees of freedom.

The $\mathrm{U}(1)^{N-1}$ theory has a lattice of allowed electric and magnetic charges, q^μ and h_μ. Generically the bare masses break the flavor symmetry $\mathrm{U}(N_f) \to \mathrm{U}(1)^{N_f}$, so states have associated quark number charges $n^j \in \mathbb{Z}$. A small $\mathcal{N} = 2$ multiplet with quantum numbers q^μ, h_μ, and n^j has a mass given by the central charge formula [2] $M = |a_\mu q^\mu + a^\mu_D h_\mu + m_j n^j|$.

The physics described by the $\mathrm{U}(1)^{N-1}$ effective theory is invariant under an $\mathrm{Sp}(2N-2;\mathbb{Z}) \quad \mathbb{Z}^{N_f}$ group of duality transformations, which acts on the scalar fields and their duals, as well as the electric, magnetic, and quark number charges, in such a way as to leave the central charges invariant. Encircling a singular submanifold may produce a nontrivial duality transformation.

More explicitly, consider $\mathbf{S} \in \mathrm{Sp}(2N-2,\mathbb{Z})$ and $\mathbf{T}$ a $(2N-2)\times N_f$ integer matrix. Then a duality transformation $(\mathbf{S},\mathbf{T})$ acts on the fields and charges as $\mathbf{a} \rightarrow \mathbf{S}\cdot\mathbf{a} + \mathbf{T}\cdot\mathbf{m}$, $\mathbf{h} \rightarrow {}^t\mathbf{S}^{-1}\cdot\mathbf{h}$, and $\mathbf{n} \rightarrow -\mathbf{T}\cdot\mathbf{h} + \mathbf{n}$. Here we have defined the column vectors ${}^t\mathbf{a} \equiv (a_D^\mu, a_\nu)$, ${}^t\mathbf{m} \equiv (m_j)$, ${}^t\mathbf{h} \equiv (h_\mu, q^\nu)$, and ${}^t\mathbf{n} \equiv (n^j)$.

In a vacuum with massless charge particles, the U(1) gauge fields that couple to them will flow to zero coupling in the infrared and will be well described by perturbation theory. Thus a one-loop calculation suffices to determine the monodromy around a submanifold of such vacua where one dyon with charges $(\mathbf{h},\mathbf{n})$ is massless:

$$\mathbf{S} = \mathbb{1} + \mathbf{h}\otimes{}^t(\mathbf{I}\cdot\mathbf{h}), \quad \mathbf{T} = \mathbf{n}\otimes{}^t(\mathbf{I}\cdot\mathbf{h}), \tag{2}$$

where $\mathbf{I} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the symplectic metric.

Solving the theory.—Our aim is to determine the analytic prepotential $\mathcal{F}$ of the low-energy Abelian theory everywhere on moduli space. We will assume, as in [1–4], that the effective coupling $\tau^{\mu\nu}(s_\ell)$ is the period matrix of a genus $N-1$ Riemann surface $\Sigma(s_\ell)$ varying holomorphically over moduli space. We solve the theory by constructing this family of surfaces.

Concretely, the construction requires a family of curves as above and a meromorphic one-form $\lambda(s_\ell)$ on $\Sigma(s_\ell)$ with

$$\frac{\partial\lambda}{\partial s_\ell} = \omega_\ell + df_\ell, \qquad \ell = 2,\ldots,N, \tag{3}$$

where ω_ℓ is a basis of holomorphic forms on Σ, and f_ℓ are arbitrary functions. The residues of λ are constrained to be integral linear combinations of the m_j. Choosing a basis of $2N-2$ one-cycles (α^μ,β_ν) on Σ with the standard intersection form $\langle\alpha^\mu,\beta_\nu\rangle = \delta^\mu_\nu$, $\langle\alpha^\mu,\alpha^\nu\rangle = \langle\beta_\mu,\beta_\nu\rangle = 0$, the VEVs of the scalar fields and their duals are defined by $a_D^\mu = \oint_{\alpha^\mu}\lambda$ and $a_\nu = \oint_{\beta_\nu}\lambda$. This defines the VEVs up to duality transformations $(\mathbf{S},\mathbf{T})$, an ambiguity corresponding to our freedom to choose a symplectic basis $(\mathbf{S})$ and to shift the winding numbers of the cycles around each of the poles $(\mathbf{T})$. The condition on the residues of λ guarantees that the correct action of $\mathbf{T}$ on the VEVs is realized. Physically, the $\mathbf{T}$ ambiguity in the periods corresponds to the freedom to shift the global quark-number current by a multiple of a U(1) gauge current [2].

We will further assume, as in [3,4], that Σ is a hyperelliptic Riemann surface with polynomial dependence on the coordinates s_ℓ and the masses m_j. A curve $y^2 = \wp(x)$, where $\wp(x)$ is a polynomial in x of degree $2N$, describes a hyperelliptic Riemann surface of genus $N-1$ as a double-sheeted cover of the x plane branched over $2N$ points.

We now proceed to determine the curve and one-form for the $N_f = 2N$ theory using this *ansatz*. First consider the SU($2N$) theory with $2N$ flavors. This theory has a dynamically generated scale that we call Λ. Let us denote the coordinates on the moduli space by the usual symbols with tildes. Recall that in this theory the $\mathrm{U}(1)_R$ symmetry is broken by instantons to a $\mathbb{Z}_{2N}$ under which $\tilde\phi_a$, $\tilde m_j$, and Λ all have charge 1. As in [3,4], we choose charges 1 and $2N$ for x and y so that this is a symmetry of the curve. Then $\wp$ must be a homogeneous polynomial in x, $\tilde\phi_a$, $\tilde m_j$, and Λ, of degree $4N$. Since Λ must enter with its instanton weight Λ^{2N}, we can write $\wp = \tilde P - \Lambda^{2N}\tilde Q + \Lambda^{4N}\tilde R$, where $\tilde P$ is of degree $4N$ in x, $\tilde\phi_a$, and $\tilde m_j$; $\tilde Q$ is of degree $2N$; and $\tilde R$ is a constant.

In the weak coupling limit $\Lambda \rightarrow 0$, the branch points of the curve are at the zeros of $\tilde P$. When two of these coincide, a cycle of the Riemann surface degenerates, corresponding to some charged state becoming massless. Such massless states appear at weak coupling wherever two of the eigenvalues of $\tilde\phi$ coincide; so $\tilde P$ has double zeros in x for these values of $\tilde\phi$. Since the $\tilde\phi_a$'s can only enter symmetrically (for gauge invariance), we must have $\tilde P = \hat P\prod_{a=1}^{2N}(x - \tilde\phi_a - \tilde\mu)$, where $\tilde\mu$ is some fixed linear combination of the $\tilde m_j$ and $\hat P$ is a homogeneous polynomial of degree $2N$. Furthermore, the whole curve must be singular as $\Lambda \rightarrow 0$, in order to produce the degeneration corresponding to the $2N$-flavor SU($2N$) beta function. Thus, at least one of the zeros of $\hat P$ must, in fact, be at $\tilde\phi_a + \tilde\mu$ for some a. Then, by symmetry in the $\tilde\phi_a$'s and using our freedom to rescale y and shift and rescale x, we find $\tilde P = \prod_{a=1}^{2N}(x - \tilde\phi_a)^2$.

Now let us break $\mathrm{SU}(2N) \rightarrow \mathrm{SU}(N)\times\mathrm{SU}(N)'\times\mathrm{U}(1)$ at a large mass scale and tune the bare masses so that there are $2N$ light flavors transforming as $(\mathbf{N},\mathbf{1},0)$. This is achieved by setting $\tilde\phi_a = M + \phi_a$ for $a = 1,\ldots,N$, and $\tilde\phi_a = -M + \phi'_a$ for $a = N+1,\ldots,2N$, with $\sum_a\phi_a = \sum_a\phi'_a = 0$. We should also set $\tilde m_j = -M + m_j$ to get quarks with masses m_j in the first SU(N) factor. In the limit $M \gg (\phi_a,\phi'_a,m_j)$ the three factors decouple. To obtain the SU(N) factor with $2N$ flavors at finite τ, we should send $\Lambda \rightarrow \infty$ such that $(\Lambda/M)^{2N} \propto q \equiv e^{i\pi\tau}$. Here we have used the one-loop renormalization group matching.

$\mathcal{N} = 2$ supersymmetric theories have no higher-loop corrections; however, nonperturbative (instanton) corrections may modify the matching by higher powers of q. Similarly, the classical mass matching may be modified by quantum corrections. Thus the matching conditions in the limit $M \rightarrow \infty$ are $(\Lambda/M)^{2N} = f(q) \propto q + \mathcal{O}(q^2)$ and $\tilde m_j = -M + g(q) + h(q)\mu_j$, with $g(q), h(q) = 1 + \mathcal{O}(q)$. The two undetermined functions g and h arise from the two possible renormalizations of the mass term: one for the flavor singlet masses and one for the flavor adjoint masses. Their leading behavior in q comes from comparing to the weak-coupling limit, $q \rightarrow 0$.

We now take the limit $M \rightarrow \infty$ in the SU($2N$) curve with $2N$ flavors. After shifting $x \rightarrow x + M$, the polynomial $\tilde P(x)$ factorizes into a piece with $2N$ zeros near $x = 0$ (relative to the scale M) and another piece with zeros all near $x = 2M$. The whole curve $y^2 = \tilde{P} + \cdots$ should factorize in this way to correspond to the decoupling low-energy sectors. Up to a conformal transformation, this describes the degeneration of the original Riemann surface into two Riemann surfaces, corresponding to the two decoupled SU(N) factors, connected by two long necks. Taking $x \ll M$, rescaling $y \to (2M)^N y$, and defining $P \equiv \prod_{a=1}^{N}(x - \phi_a)$, the curve becomes approximately

$$y^2 = P^2 - f\tilde{Q}(x + M, \tilde{m}_j, \tilde{\phi}_a) + f^2 M^{2N}\tilde{R}. \quad (4)$$

The factorization discussed above means this polynomial is independent of the ϕ'_a. Being a symmetric function of the $\tilde{\phi}_a$, $\tilde{Q}$ must, in fact, have no $\tilde{\phi}$ dependence. Factorization also requires that the branch points of (4) be at $|x| \ll M$, so the coefficients of positive powers of M must vanish identically in x and q. Thus $\tilde{R} = 0$, while in terms of $g(q)\mu = \tilde{\mu} + M$ and $h(q)\mu_j = \tilde{\mu}_j$, $\tilde{Q}$ must take the form $\tilde{Q}(x + g\mu, h\mu_j)$.

To further constrain the curve we construct the meromorphic form λ. A basis of holomorphic one-forms on our hyperelliptic curve are $\omega_\ell = x^{N-\ell}dx/y$ for $\ell = 2, \ldots, N$. Noting that $P = \sum_{\ell=0}^{N} s_\ell x^{N-\ell}$, one can integrate the differential equation (3) to find $\lambda \propto \ln[(P - y)/(P + y)]dx$, which has logarithmic singularities at $x = \epsilon_j$, the zeros of $\tilde{Q}$. These logarithms can be converted into poles by adding the total derivative $d[(x + b)\ln((P + y)/(P - y))]$ to λ. This does not affect the differential equation (3). The resulting form has poles $\pm(\epsilon_j + b)dx/(x - \epsilon_j)$ at the two preimages of ϵ_j.

The requirement that the residues of λ be linear in the quark masses implies that the ϵ_j, the zeros of $\tilde{Q}$, are linear in the masses. The most general flavor-symmetric $\tilde{Q}$ with this property is $\tilde{Q} = \prod_j(x + g\mu + h\mu_j)$. In fact, one renormalization of the masses, say, $h(q)$, can be absorbed into the definition of the coupling; henceforth we will set $h = 1$. Finally, demanding that the residues of λ be the bare quark masses $\pm m_j$ implies $b = (g - 1)\mu$.

The resulting one-form has an additional pole at $x = \infty$, which is easily calculated to be $2N\mu g(1 - f)^{-1/2}dx/x$. As we will show below, the residues of λ on a given sheet of the curve must sum to zero in order to reproduce a scale-invariant theory. Since $2N\mu = \sum_j m_j$, the residue sum vanishes only if $f = 1 - g^2$.

The curve is thus

$$y^2 = \prod_{a=1}^{N}(x - \phi_a)^2 - (1 - g^2)\prod_{j=1}^{2N}(x + g\mu + \mu_j), \quad (5)$$

and the one-form is

$$\lambda = \frac{x + (g - 1)\mu}{2\pi i} d\left[\ln\left(\frac{\prod_a(x - \phi_a) - y}{\prod_a(x - \phi_a) + y}\right)\right]. \quad (6)$$

To finish the argument we need to find the function $g(q)$. In principle, g can depend on N as well as q. We determine first its N dependence by induction on N, then its q dependence by matching onto the $N = 2$ solution.

The induction proceeds by considering the breaking of SU(N) with $N_f = 2N$ down to SU($N - 1$) $\times$ U(1) with $2N - 2$ light flavors transforming as $(\mathbf{N} - \mathbf{1}, 0)$. Set

$$\phi_a = \begin{cases} M + \phi'_a, & a = 1, \ldots, N - 1, \\ (1 - N)M, & a = N, \end{cases}$$

$$m_j = \begin{cases} -t(q)M + m'_j, & j = 1, \ldots, 2N - 2, \\ -u(q)M + v(q)\mu', & j = 2N - 1, 2N. \end{cases} \quad (7)$$

The limit $M \to \infty$, keeping ϕ'_a and m'_j fixed, achieves the desired breaking at weak coupling if $t(0) = 1$ and $u(0) \neq 1$. The functions t, u, and v can be determined by demanding that our curve (5) reduces to a curve of the same form with $N \to N - 1$. Plugging (7) into (5) and taking the $M \to \infty$ limit, one finds after some algebra that $g_{N-1} = g_N$. The one-loop renormalization group matching condition that $1 - g_N(q)^2 \propto q$ independent of N implies that the bare couplings satisfy $\tau_N = \tau_{N-1}$ at weak coupling.

We determine the unknown factor $g(q)$ by matching to the solution [2] of the SU(2) theory with 4 massless flavors $\tilde{y}^2 = \prod_{i=1}^{3}(\tilde{x} - e_i s_2)$, where $3e_1 \equiv \theta_2^4 + \theta_3^4$, $3e_2 \equiv -\theta_3^4 - \theta_1^4$, and $3e_3 \equiv \theta_1^4 - \theta_2^4$, as defined in [2]. This curve is equivalent to (5) with $\mu = \mu_j = 0$ if there is an SL(2, $\mathbb{C}$) transformation relating the x and $\tilde{x}$ coordinates which maps the branch points of one curve into those of the other. This condition determines g in terms of the e_i up to permutations, giving six possible solutions, of which only two have the correct weak-coupling asymptotic form $g = 1 + \mathcal{O}(q)$:

$$g = \frac{\theta_2^4 + \theta_1^4}{\theta_2^4 - \theta_1^4} \quad \text{or} \quad g = \frac{\theta_3^4 - \theta_1^4}{\theta_3^4 + \theta_1^4}. \quad (8)$$

These are physically equivalent: they are related by $\tau \to \tau + 1$, or, equivalently, by a conventional choice of the origin of the θ angle. This completes the determination of the $N_f = 2N$ curve. The curves for $N_f < 2N$ are obtained by taking appropriate limits as $q \to 0$ and $m_j \to \infty$. In particular, this reproduces the results of [3,4].

Properties of the solution.—The modular functions (8), and therefore the $N_f = 2N$ curve, are invariant under $T^2 : \tau \to \tau + 2$. This is expected, since it corresponds to a shift in the theta angle $\theta \to \theta + 2\pi$, which is a symmetry of the theory. In addition, there is a strong-weak coupling duality. Taking, for example, the first solution in (8), we see that $g \to -g$ under $S : \tau \to -1/\tau$. The curve (5) is left invariant if, at the same time, we take $\mu \to -\mu$ and $\mu_j \to +\mu_j$, verifying a conjecture of Ref. [5].

As a check on the validity of our solution, we now show that it reproduces the positions and monodromies of two classes of singularities at weak coupling.

The first class of singularities occurs whenever $\phi_a = \phi_b$, and corresponds classically to the restoration of a non-Abelian gauge symmetry. Because the β function vanishes, the semiclassical monodromies around them are actually the classical monodromies in the Weyl group of SU(N), which act by permuting the ϕ_a's. The breaking (7) implies that all the SU($N - 1$) $N_f =$

$2N - 2$ singularities and associated monodromies are reproduced by the SU(N) $N_f = 2N$ curve. This fact allows us to check the monodromies by induction in N. We need only compute for SU(N) that monodromy not contained in the Weyl group of SU($N - 1$). This is the "special monodromy" identified in [3], given in the scale-invariant theory by $\mathbf{S} = \left(\begin{smallmatrix} Q & 0 \\ 0 & P \end{smallmatrix}\right)$, where ${}^tQ^{-1} = P$ is the $(N - 1) \times (N - 1)$ matrix representation of a $(1, \ldots, N)$ cyclic permutation.

For weak coupling, $|q| \ll 1$, and VEVs much larger than the bare masses $\phi_a \gg m_i$, the curve is approximately $y^2 = \prod(x - \phi_a)^2 - qx^{2N}$. Degenerations where branch points collide occur whenever $\phi_a = \phi_b$ up to $\mathcal{O}\sqrt{q}$ corrections, corresponding to the semiclassical positions of these singularities. The special monodromies can be conveniently measured by traversing a large circle in the s_N plane, fixing the other $s_\ell = 0$, where the curve factorizes as $y^2 = [(1 - \sqrt{q})x^N + s_N][(1 + \sqrt{q})x^N + s_N]$. The branch points are arranged in N pairs with a pair each Nth root of unity times $s_N^{1/N}$. As $s_N \rightarrow e^{2\pi i}s_N$, these pairs rotate into one another in a counterclockwise sense.

Choose cuts and a basis for the cycles on the SU(N) surface as shown [for SU(3)] in Fig. 1. The intersection numbers for these cycles are $\langle \beta_a, \beta_b \rangle = \langle \gamma^a, \gamma^b \rangle = 0$ and $\langle \beta_a, \gamma^b \rangle = -\delta_a^b + \delta_a^{b-1}$. A canonical basis of cycles is then β_μ and $\alpha^\mu \equiv \sum_{a=1}^{\mu} \gamma^a$. Note that β_N is not independent of the β_μ's: a simple contour deformation shows that $\sum_a \beta_a = 0$. Similarly, $\sum_a \gamma^a = 0$. These relations hold precisely when the residues of λ on the x plane sum to zero; see the discussion preceding (5).

As $s_N \rightarrow e^{2\pi i}s_N$ the $\beta_a \rightarrow \beta_{a+1}$, a cyclic permutation. Using the fact that the β_a sum to zero, this monodromy is the matrix P acting on the β_μ cycles, and thus on the a_μ periods. Similarly, the α^μ cycles (and thus the a_D^μ periods) transform by ${}^tP^{-1}$. This is the classical monodromy predicted above.

This completes the induction step in the calculation of these monodromies. For the initial step we match our

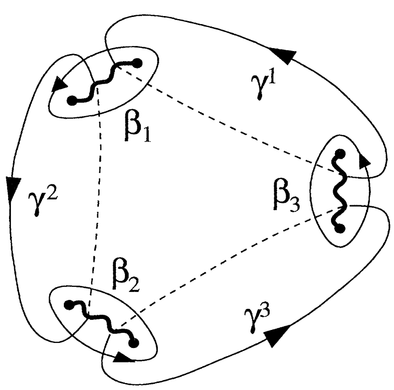

FIG. 1. Contours for a basis of cycles for the SU(3) curve. The thick wavy lines represent the cuts, solid contours are on the first sheet, and dotted ones are on the second.

solution and monodromies to the SU(2) solution found in [2]. When the bare masses are $m_i = (m, m, 0, 0)$ we find an explicit SL(2, $\mathbb{C}$) transformation relating the two curves. In addition, the discriminants of the two curves factorize in the same way for several other cases, implying that the positions and monodromies of the singularities in these cases agree. In fact, mimicking the argument of [2], one can reverse this procedure to show that these conditions fix the form of the SU(2) curve to be (5).

The second class of singularities occurs whenever $\phi_a = -m_i$, and corresponds to the $q_a^i, \tilde{q}_i^a$ hypermultiplet becoming massless. The massless quark can be taken to have electric charge 1 with respect to a single U(1) factor and to have $n^j = \delta_1^j$. The semiclassical monodromy around this singularity can be read off from (2).

Consider the curve near a classical quark singularity, say, $\phi_1 + m_1 \sim 0$. At weak coupling $g\mu + \mu_j \simeq m_j$ and for $x \sim \phi_1$ the curve is approximately $y^2 = (x - \phi_1)^2 - 4qC(x + m_1)$, where $C = \prod_{i>1}(x + m_i)/\prod_{a>1}(x - \phi_a)$ is a slowly varying function of x and s_ℓ. This has a double zero at $x = -m_1 + qC$ for $\phi_1 = -m_1 - qC$, which is indeed near the classical singularity for small q. Define the period a_1 by a contour enclosing the pole at $-m_1$ (recall that changing which poles are enclosed by a given contour corresponds to a physically unobservable redefinition of the quark number charges). One then finds that as ϕ_1 winds around the singular point the two branch points are interchanged. The monodromy that results in nontrivial only in a two-dimensional block of (2), for which we find $\mathbf{S} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathbf{T} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, in agreement with the semiclassical prediction.

We thank R. Donagi, M. Douglas, D. Friedan, K. Intriligator, R. Leigh, N. Seiberg, M. Strassler, and E. Witten for useful conversations. The work of P. C. A. is supported by NSF Grant No. PHY92-45317 and by the Ambrose Monell Foundation. M. R. P. is supported in part by NSF Grant No. PHY92-45317 and by the W. M. Keck Foundation. A. D. S. is supported in part by DOE Grant No. DE-FC02-91ER75661 and by an A. P. Sloan Fellowship.

Note added.—As this work was being completed we received Ref. [6], which addresses related problems.

[1] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994).
[2] N. Seiberg and E. Witten, Nucl. Phys. **B431**, 484 (1994).
[3] P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. **74**, 3931 (1995).
[4] A. Klemm, W. Lerche, S. Theisen, and S. Yankielowicz, Phys. Lett. B **344**, 169 (1995).
[5] R. Leigh and M. Strassler, Report No. hep-th/9503121.
[6] A. Hanany and Y. Oz, Report No. hep-th/9505075.