Reentrance Phenomena in Noise Induced Transitions

F. Castro, A. D. Sánchez, and H. S. Wio
Centro Atómico Bariloche, Comisión Nacional de Energía Atómica, and Instituto Balseiro, Universidad Nacional de Cuyo, 8400-S. C. de Bariloche, Río Negro, Argentina
(Received 19 April 1995)

We analyze a model that has been shown to undergo a purely noise induced transition, from a monostable regime to a bistable one, when it is submitted to a white or colored noise source. We show, using a consistent interpolating Markovian approximation for the colored noise case, that for large values of the correlation time, the system undergoes a new transition to a monostable state, indicating a reentrancelike phenomenon in its phase diagram. Numerical results support our findings.

PACS numbers: 05.40.+j

The study of dynamical systems subject to a noise perturbation has become a recurrent theme in physics, chemistry, and biology, as well as in several other areas. Particularly for nonequilibrium systems, where the macrovariables obey nonlinear equations of motion, noise plays a crucial role. For instance, the system can overcome potential barriers and reach different macrostates due to only the presence of noise [1]. One aspect that attracted considerable attention was the fact that some systems, when far from thermodynamic equilibrium, due to the influence of external noise sources, show the striking characteristic of undergoing transitions to new states that sometimes are not present in the deterministic description. These transition phenomena pose a fascinating problem as, contrary to all intuition, it is the environmental randomness that deeply influences the macroscopic system's properties, inducing a more structured behavior. These types of nonequilibrium transition phenomena have been called *noise induced transitions* [2,3].

On the other hand, more realistic models of physical systems require considering noise sources with finite correlation times (i.e., *colored noise*). For example, in order to describe the static and dynamical properties of dye lasers, the usual model includes in its stochastic differential equations (SDE's) not only the standard internal white noise, but also an external colored noise [4]. The effect of time correlations in the fluctuations has also been taken into account in several models [2,3,5,6]. Some recent papers and reviews on the colored noise problem [7–10] offer a view of the state of the art. Many efforts were oriented to obtaining Markovian approximations, with the aim of capturing the essential features of the original non-Markovian problem. Along this line, a recent approach was based on an interpolation procedure [11].

In this paper, we present an analysis of a chemical reaction system and/or genic selection model [2,3,5], when it is subject to a colored noise source, by using the interpolation procedure. This approach allows us, at variance with previous studies, to obtain the complete phase diagram for the whole range of parameters. The choice of the above indicated genic model is due to the fact that it is an archetype of the kind of models studied within the realm of noise induced transitions [3]. The result of our study, supported by numerical evidence, is that the phase diagram shows a novel feature corresponding to a reentrancelike phenomenon.

The colored noise problem can be represented by a general SDE of the form

$$\dot{y}(t) = f[y(t)] + \epsilon(t), \tag{1}$$

where $\epsilon(t)$ is a non-delta-correlated noise source. We can make the following considerations about the noise. First, as real noise sources reflect the cumulative effect of weakly coupled individual fluctuations, the central limit theorem states that the noise must be Gaussian. Second, although the complete system that we are taking into account is non-Markovian, it is usual practice to choose the noise source to be Markovian. This is not the most general noise that we can consider, but it gives a reasonable representation of many physical processes [3,7–9]. It has also the advantage of reducing the mathematical complexity. With those assumptions and the requirement of stationarity, there is only one choice, namely, the Ornstein-Uhlenbeck process. In addition, this selection yields the right white noise limit and allows us to correlate our results with previous ones. As usual, we assume that it has zero mean, and correlation

$$\langle \epsilon(t)\epsilon(t')\rangle = \frac{D}{2\tau}\exp\left[-\frac{|t-t'|}{\tau}\right]; \tag{2}$$

here D denotes the noise intensity and τ is the correlation time.

The interpolation scheme [11] arises from a path integral point of view through the consideration of the exact Lagrangian associated with the non-Markovian process represented by Eq. (1) [10]. In the limits $\tau \rightarrow 0$ (white noise) or $\tau \rightarrow \infty$, the problem reduces to Markovian forms, and it is possible to obtain the Fokker-Planck type Lagrangians associated with each case. The aim of the interpolation procedure is to get a Lagrangian of a Markovian Fokker-Planck form, which becomes exact in the above mentioned limits, offering a reasonable description for the intermediate correlation time regime. This is accomplished by considering a Lagrangian with an interpolation function that fulfills particular

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limit conditions assuring that the exact Lagrangians are recovered in the limits of white noise and very large correlation time. The Markovian SDE and the Fokker-Planck (Stratonovich sense) equation associated with this interpolating Lagrangian are

$$\dot{y} = f(y)\theta[\tau f'(y)] + \sqrt{D}\theta[\tau f'(y)]\xi(t), \tag{3}$$

$$\begin{aligned}\frac{\partial[P_I(y,t)]}{\partial t} = &-\frac{\partial}{\partial y}\Big(\{f(y)\theta[\tau f'(y)] \\ &+ D\theta[\tau f'(y)]\theta'[\tau f'(y)]\}P_I(y,t)\Big) \\ &+ D\frac{\partial^2}{\partial y^2}\Big(\theta^2[\tau f'(y)]P_I(y,t)\Big),\end{aligned} \tag{4}$$

where $\xi(t)$ is a white noise source $[\langle \xi(t)\rangle = 0$, $\langle \xi(t)\xi(t')\rangle = 2\delta(t-t')]$ and $\theta[\tau f'(y)]$ is the interpolating function, fulfilling the limit properties

$$\lim_{\tau\to 0} \theta[\tau f'(y)] = 1, \tag{5}$$

$$\lim_{\tau\to\infty} \theta[\tau f'(y)] = -[\tau f'(y)]^{-1}. \tag{6}$$

The model equation that we consider in this work is

$$\dot{x} = \frac{1}{2} - x + \lambda x(1-x) + x(1-x)\epsilon(t), \tag{7}$$

which follows from a particular genic model [5] or from a chemical reaction [2,3], where $\epsilon(t)$ is an Ornstein-Uhlenbeck noise.

In order to apply the interpolation procedure, we make the change of variables

$$y = \ln\Big(\frac{x}{1-x}\Big), \tag{8}$$

so that the original multiplicative SDE (7) becomes the SDE with additive noise

$$\dot{y} = -\sinh(y) + \lambda + \epsilon(t). \tag{9}$$

For the sake of mathematical simplicity, we consider the so-called symmetric case, that is, we assume $\lambda = 0$. Hence, by choosing the family of interpolating functions [11]

$$\theta[\tau, y] = \frac{1 + c[\tau\cosh(y)]}{1 + c[\tau\cosh(y)]^2}, \tag{10}$$

where c is the parameter that generates the different functions, it is possible to write the Fokker-Planck equation (4) and from it to obtain the following stationary probability distribution in terms of the original variable x:

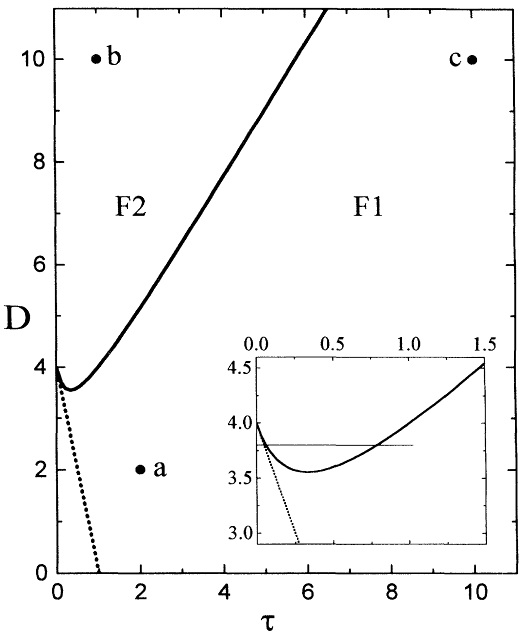

FIG. 1. Phase diagram of the system. The full line is the interpolation result and the dotted line the small correlation time result of Refs. [3,5]. $F1$ denotes the monostable phase and $F2$ the bistable one. The stationary distributions corresponding to values of D and τ denoted by the points a, b, c are shown in Fig. 2. The inset exhibits in detail the reentrance zone; the horizontal thin line corresponds to the path taken in Fig. 3.

$$\begin{aligned}p_{st}(x) = &\frac{N}{x(1-x)}\frac{[4x^2(x-1)^2 + c\tau^2(1-2x+2x^2)^2]}{[4x^2(x-1)^2 - 2c\tau x(x-1)(1-2x+2x^2)]} \\ &\times \exp\left\{\frac{2}{D}\left[\frac{1}{2cx(1-x)} - \frac{\tau(1-2x+2x^2)^2}{8x^2(x-1)^2} - \frac{1+c}{c^2\tau}\ln\left(1 - c\tau + \frac{c\tau}{2x(1-x)}\right)\right]\right\},\end{aligned} \tag{11}$$

where N is a normalization constant.

In the absence of noise, the deterministic equation associated with (7) can be exactly solved, rendering a solution that presents only one stable equilibrium point at $x = 1/2$. In contrast, when a noise source is present, depending on the values of noise intensity as well as on the correlation time, the stationary distribution presents an isolated maximum at $x = 1/2$ [the monostable phase ($F1$)] or two maxima located symmetrically around $x = 1/2$ [the bistable phase ($F2$)]. The transition from a monostable to a bistable state is solely produced by the existence of the noise source [2,3]. In previous works, the existence of this transition was recognized, although the studies were carried out only in the limit of small correlation times. However, our approximation allows us to obtain the complete phase diagram. The transition line, indicated in Fig. 1, is given by

$$D = \frac{4(1 + 2\tau + \tau^2)}{1 + 3\tau}. \tag{12}$$

This result gives the exact white noise transition point, and the particular value of the interpolation parameter, chosen to be $c = -1$, renders a transition line whose slope at the origin agrees with those obtained in previous works in the small correlation time regime [2,3,5]. It is worth remarking that, with this value of c, the interpolation procedure coincides with the UCNA (Unified Colored Noise Approximation) [12].

Together with the results coming from the interpolation procedure, we have performed computer simulations of the Markovian system of equations equivalent to Eq. (7) in order to obtain the stationary distribution for the points a, b, c indicated in Fig. 1, and to verify the behavior predicted in the phase diagram [Fig. 2(a)–2(c)].

The phase diagram makes it apparent that, for some regions of values of noise intensity ($32/9 < D < 4$) and for zero and small values of correlation time, the system is found in a monostable phase ($F1$). After increasing the correlation time beyond some threshold value, the system undergoes a transition to a bistable phase ($F2$). However,

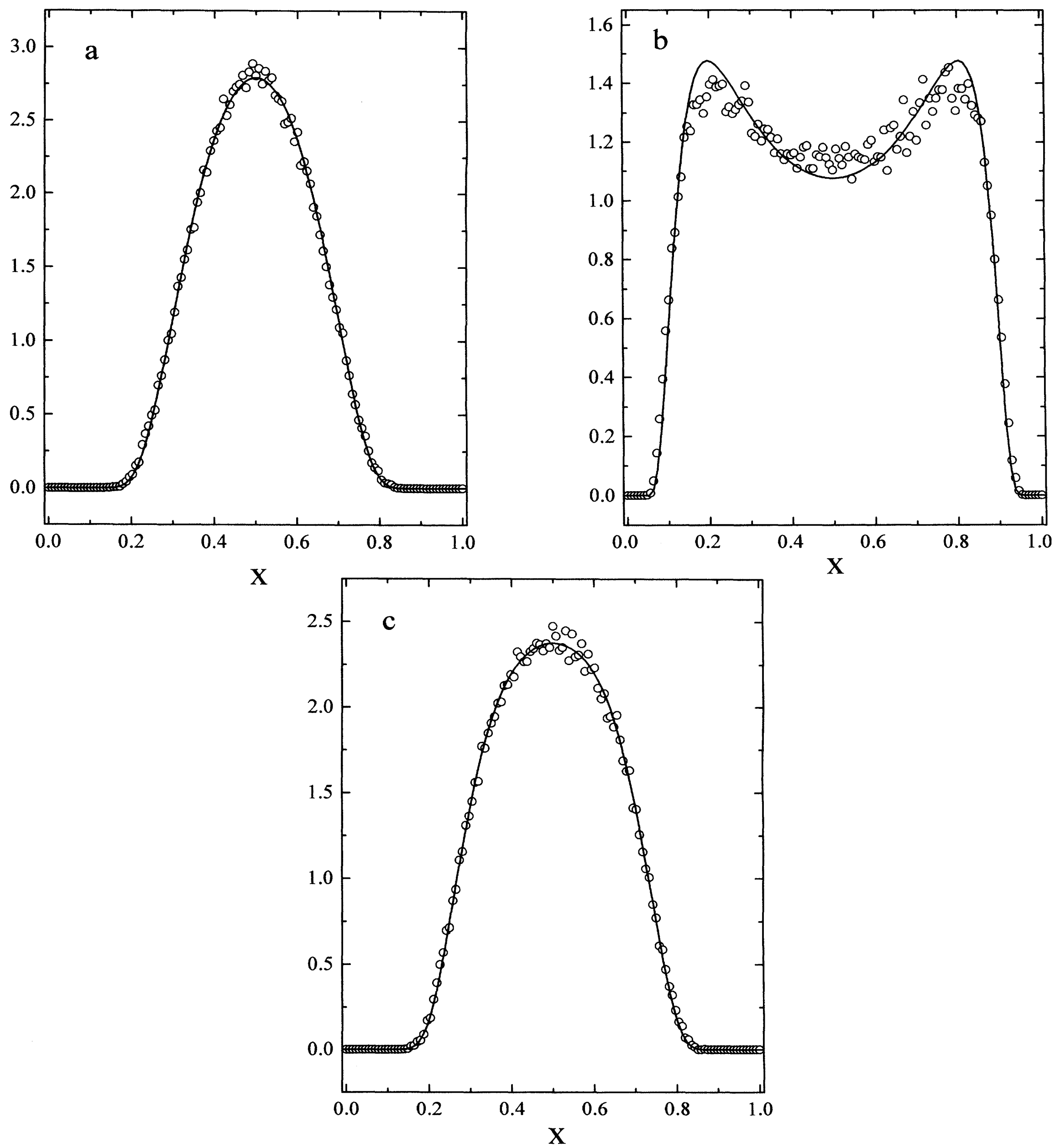

FIG. 2. Stationary distribution for different values of the parameters. The circles are the results of the simulations (100 000 realizations); the full line corresponds to the interpolation result. (a) $D = 2, \tau = 2$ (monostable); (b) $D = 10, \tau = 1$ (bistable); (c) $D = 10, \tau = 10$ (monostable).

if we further increase the correlation time, the system goes back to a monostable phase ($F1$). This remarkable effect of reentrance can be seen in Fig. 3, where we have depicted only the tops of the stationary distribution in order to make apparent the transition phenomenon (as we are looking at a parameter zone near the bottom of the transition line, the existence of several maxima in the distribution is only visible in this way). We interpret this reentrancelike result as a manifestation that, in the long correlation time limit $\tau \rightarrow \infty$, we essentially recover the deterministic behavior [9]. At this point it is worth noting that a related reentrance effect, albeit in a different context, has been recently found in Ref. [13]. There the authors studied a spatially extended system (while our system is zero dimensional), and only white noise was considered, rendering a true phase transition.

The present result, supported by numerical evidence, confirms the capacity of the interpolation scheme to capture the essential features of colored Ornstein-Uhlenbeck noise within a tractable Markovian approximation.

A question to be raised is if similar peculiarities can be found in other models submitted to such a noise source or for other colored noise sources. This problem will be the subject of further work. However, the most relevant question is if this phenomenon will happen in real physical systems. We expect that the present result will stimulate the experimental search for this effect.

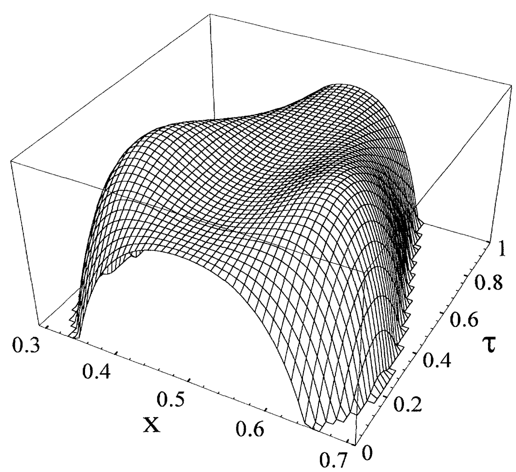

FIG. 3. Reentrance effect. Plot of the top of the stationary distribution obtained with the interpolation scheme (arbitrary normalization) as a function of the correlation time. The noise intensity is $D = 3.8$. It is possible to see the succession of phases $F1 \rightarrow F2 \rightarrow F1$ as the correlation time is increased.

The authors thank G. Marshall for providing the program to perform the numerical work, and V. Grunfeld for a critical reading of the manuscript. Partial support from CONICET, Argentina, through Grant No. PID 33366/92 is also acknowledged.

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