Boltzmann Fluctuations in Numerical Simulations of Nonequilibrium Lattice Threshold Systems

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Nonequilibrium threshold systems such as slider blocks are now used to model a variety of dynamical systems, including earthquake faults, driven neural networks, and sliding charge density waves. We show that for general mean field models driven at low rates fluctuations in the internal energy field are characterized by Boltzmann statistics. Numerical simulations confirm this prediction. Our results indicate that mean field models can be effectively treated as equilibrium systems.

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Slider block models [1–5] are simple examples of driven nonequilibrium threshold systems on a lattice. In addition to simulating the aspects of earthquakes and frictional sliding, these models may also represent the dynamics of neurological networks [6] and sliding charge density waves [7]. Given the importance of these systems, a basic question is whether any of these nonequilibrium lattice models exhibit similarities to equilibrium systems. If these systems possess any kind of stable, time-averaged energy distribution function, standard techniques and methods of equilibrium statistical mechanics might then be available for use in analysis of simulation results and interpretation of system dynamics.

To our knowledge, previous work on these models has focused almost exclusively on the critical phenomena associated with clusters of failed points. Scaling in the cluster numbers has been compared directly to the scaling observed in the Gutenberg-Richter magnitude frequency relation for earthquakes. It is known [8], however, that critical phenomena observed with percolation clusters in Ising models need have no relation to critical behavior of the magnetization. A percolation transition of clusters in an Ising model can be observed at infinite temperature, even though the correlation length of magnetic fluctuations is small or even zero. A fundamental question is whether scaling of cluster numbers in slider block models is related to critical phenomena in the underlying order parameter, which is the lattice-averaged stretch of the loading springs (the slip deficit). A relationship between critical phenomena in the clusters and critical phenomena in the slip deficit might then be established, if it could be demonstrated that fluctuations in the driven lattice models are isomorphic to the Boltzmann fluctuations characterizing equilibrium systems.

To summarize our main result: We have found broad classes of lattice models that possess stable energy distributions. For nearest neighbor slider block simulations, the block energy distribution is a generalized Boltzmann function as the model approaches mean field, where fluctuations are minimal.

Consider a typical equilibrium system for which $E_T \approx$ const, except for small fluctuations [9], the mean square probability of which decreases in magnitude as $1/\sqrt{N}$, where N is the number of particles (or degrees of freedom or modes). The internal energy E_i of each independent field variable (molecules, spins, etc.) executes small fluctuations about the time-averaged mean energy. Assuming that the system obeys the postulate of equal *a priori* probability, the method of most probable distributions [9] can then be used to show that the expected distribution of block energies is closely related to a Boltzmann distribution. We begin by dividing the possible energy states into $\{q = 1, \ldots, Q\}$ energy bands $E(q)$ occupied with probability $p(q)$:

$$\sum_q p(E_q) = 1\,, \qquad \sum_q p(E_q)E_q = E_T \approx \text{ const}. \tag{1}$$

Most slider block models oscillate around a fixed value of energy. Consider a simple cellular automaton (CA) model [2,5], in which N massless blocks are connected by a network of nearest neighbor coupling springs K_C, and to a loader plate by a spring of constant K_L. The loader plate translates at a velocity V, increasing the

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force on each block as time passes. Block i possesses both a force threshold σ^F and a nominal residual force σ^R. Blocks move when the force on the ith block σ_i equals or exceeds the threshold. Farther neighbor models can also be implemented to represent elastic continua [3] using long range coupling springs whose spring constants decay with distance r as $1/r^3$. An advantage of massless CA models is that the dynamics of large N models can be examined on modest workstations, which can be important when correlation lengths are large and finite size effects are important.

The Hamiltonian [3,4] for a slider block model with arbitrary range interactions having coupling springs of fixed strength K_C and driven at velocity V can be written in the form

$$H(\phi,\phi) = (1/2)\sum_i \left\{ K_L(\phi_i - V\tau)^2 + (1/2)K_C \sum_{j=\text{int}} [\phi_j - \phi_i]^2 \right\}, \qquad (2)$$

where K_T is the total spring constant ($K_T = K_L + 2dK_C$, d is the dimension of space). The sum over i is over all sites in the lattice, and the sum over j is over all interacting blocks within the range of interaction R but excluding site i. The slip deficit $\phi_i(t) = s_i(t) - Vt$; $s_i(t)$ is the slip of block i at time t. In (2), the modification $\phi_i \rightarrow \phi_i - V\tau$ from the usual slider block Hamiltonian [3] has been introduced [10] for dynamical simulations in which a nonzero loader plate time scale τ exists. Physically, the plate must move to store energy in the system before slip of a block can occur. The force (stress) σ_i on block i is

$$\begin{aligned}\sigma_i &= -\partial H/\partial \phi_i \\ &= -\left\{ K_L\phi_i + K_C \sum_{j=\text{int}} [\phi_j - \phi_i] \right\} + K_L V\tau. \qquad (3)\end{aligned}$$

The first terms in brackets describe the force in the springs at time t_j the last term is the stress that is produced on one loader plate update and can be thought of as a "prestress." In the following, we specialize for models with $V \rightarrow 0$, in which only one avalanche occurs following each loader plate update.

For CA models, a rule to generate the dynamics must be specified. The simplest example is the modified Mohr-Coulomb friction law, in which each block has a static failure threshold σ^F, and a residual stress σ^R at which the block sticks. The basic dynamical equation is

$$s_i(t+1) = s_i(t) + J(\sigma_i)\Theta(\sigma_i - \sigma^F), \qquad (4)$$

where $\Theta(x)$ is a Heaviside step. This jump function $J(\sigma_i)$ can be either deterministic or stochastic. Previous work has focused primarily on deterministic models. Examples [2,3,5,11–13] of deterministic jump functions include

$$\begin{aligned} J_1 &= (\sigma_i - \sigma^R)/K_T, \\ J_2 &= (\sigma^F - \sigma^R)/K_T, \end{aligned} \qquad (5)$$

where σ^R is a residual stress. This expression is valid also for a model with longer range springs (interactions), in which each block interacts with N_R other blocks via springs with spring constants K_C. Each block jumps from its current stress at failure to the position having the specified residual stress σ^R, thus (4) and (5) are examples of a deterministic rule.

Recent work has shown that models using the J_2 jump function give rise to periodic behavior [12,13]. We expect that deterministic models using the J_1 jump, but with weak coupling, will also be periodic. In order to satisfy the postulate of equal *a priori* probability, the system must be allowed to explore phase space. We focus on models that have a stochastic character, and one of these [14] is used here. For this model, the jump is

$$J_s = J_1(1 - W\rho), \qquad (6)$$

where $0 \leq W \leq 1$ is a noise amplitude, and ρ is a uniformly distributed random number on $\rho \in [0,1]$.

The slip deficit $\phi_i(t)$ of a block fluctuates around a time-averaged value $\eta_i = \overline{\phi_i}$. The fluctuating part $\psi_i(t)$ is defined by $\phi_i(t) = \eta_i + \psi_i(t)$. The Hamiltonian (2) with $V \rightarrow 0$ can be written as $H(\phi,\phi) = H_0(\eta,\eta) + H_1(\eta,\psi) + H'(\psi,\psi)$. Taking the time average of H, we observe that $\overline{H_0} = H_0$, $\overline{H_1} = 0$, $\overline{H'} \neq 0$. H_0 is a constant, H_1 executes small fluctuations about 0, and H' fluctuates about a nonzero value. Our interest is in the nonzero time-averaged occupation numbers n_q for the various energy bands centered on E_q, so we focus attention on H' and define $\overline{H'} \equiv \sum_q n_q E_q$. Assuming that all configurations have equal *a priori* probability, the probability density function (PDF) of individual springs is Gaussian in ψ. The total energy of ν identical independent springs [15] is then χ^2 distributed with ν degrees of freedom. In particular, the energy in a pair of springs is Boltzmann distributed. The PDF for our model in mean field can then be obtained as the χ^2 distribution in total energy E_q with 4 degrees of freedom.

As in the Boltzmann transport theorem from kinetic theory [9], we expect the property of microscopic chaos will exist in slider block models when the interaction between neighboring blocks is large enough to self-organize the blocks against the competing stochastic noise from the block jumps. Some level of stochastic noise must be present, particularly for small values of K_C, to prevent the blocks from phase locking into a limit cycle [12,13]. Since $R = (K_C/K_L)^{1/2}$ is a measure of the range of interaction, models with large values of R display mean field characteristics. Mean field models with a given noise level are more likely to demonstrate Boltzmann statistics: These models are associated with decreasing amplitude fluctuations at all but the largest wavelengths, thus the assumption that $\sum_q n_q E_q = $ const is more likely to be valid.

We have carried out a number of stochastic simulations on a square lattice with nearest-neighbor interactions using (6). For convenience we normalize the time-averaged fluctuation $\psi_i(t)$ to unity by defining the variance ω_i^2 of the fluctuation $\omega_i^2 = \overline{\psi_i^2}$, so that $\psi_i'(t) = \psi_i(t)/\omega_i$. On an infinitely large lattice, $\eta_i = \eta =$ const, $\omega_i = \omega =$ const, but the presence of finite boundaries causes η_i and ω_i to vary spatially. We therefore accumulate time-averaged statistics using the normalized energies $H_i''(\psi_i', \psi_j')$ to construct the time-averaged, cumulative distribution function for the block energies, and to plot the lattice-averaged energy against time. Operationally, we define 10 000 energy bands centered on each E_q ($q = 1, \ldots, 10\,000$) and, upon termination of all block motion, count the number of block energies falling into each narrow energy band following a loader plate update. Defining $p(E_q) = n_q/N$, the probability of a block being in the energy band centered on E_q is $E_q p(E_q)$, the χ^2 distribution [15] with $\nu = 4$. Since $p(E_q) = (1/\varepsilon)\exp[-E_q/\varepsilon]$, the cumulative distribution function (CDF) $P(E')$ is obtained:

$$P(E') = \int_0^{E'} E' \exp[-E']dE' = 1 - (1 + E')\exp[-E'], \qquad (7)$$

where $E' = E/\varepsilon$. Defining the lattice average of the block energies H_i'' at fixed time t by $\langle H''(t)\rangle$, we obtain the time-averaged energy per block ε (= "temperature") as $\varepsilon = (1/2)\overline{\langle H''(t)\rangle}$. With ε fixed, Eq. (7) represents a prediction, with no free parameters, of the energy distribution obtained from simulation data.

Two examples of our simulations are given in Figs. 1 and 2. Simulations were carried out on a 100×100 square lattice of points, and in all figures $K_L = 1$. In Fig. 1, $K_C = 1$, $W = 0.8$, and in Fig. 2 $K_C = 50$, $W =$

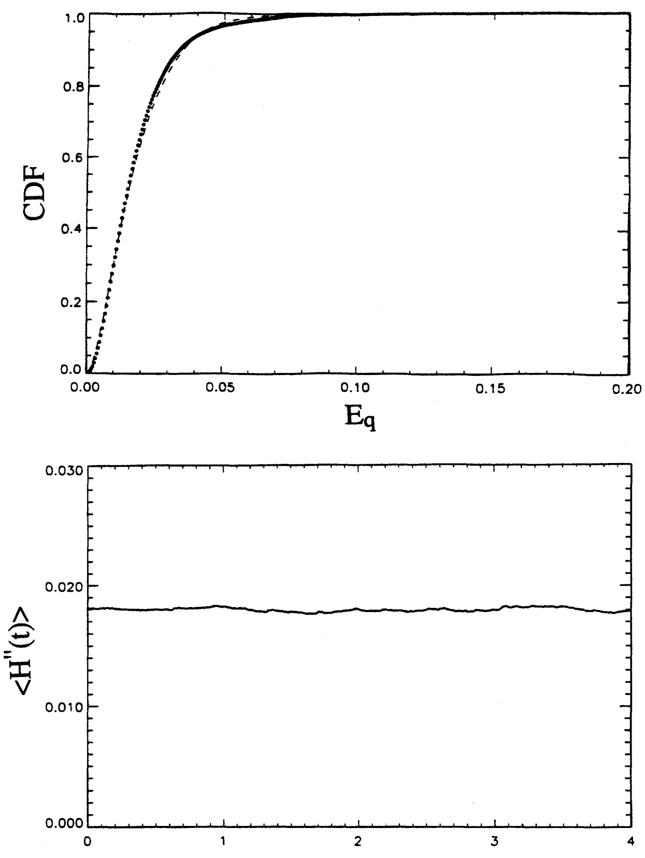

FIG. 1. (Upper) Cumulative distribution function (CDF) of block energies E_q from our simulation (dots) and prediction (dashed line) using (7). (Lower) Lattice average of the block energies $\langle H''(t)\rangle$ as a function of time. Mean energy ε used in constructing the dashed line in the upper figure is obtained by averaging $\langle H''(t)\rangle$ over the time interval of the simulation, the average value obtained being 2ε. Parameters in this simulation are $K_C = 1$, $K_L = 1$, and $W = 0.8$. Time-averaged energy $\varepsilon = 3.605$, $\sigma^F = 35$, and $\sigma^R = 0$.

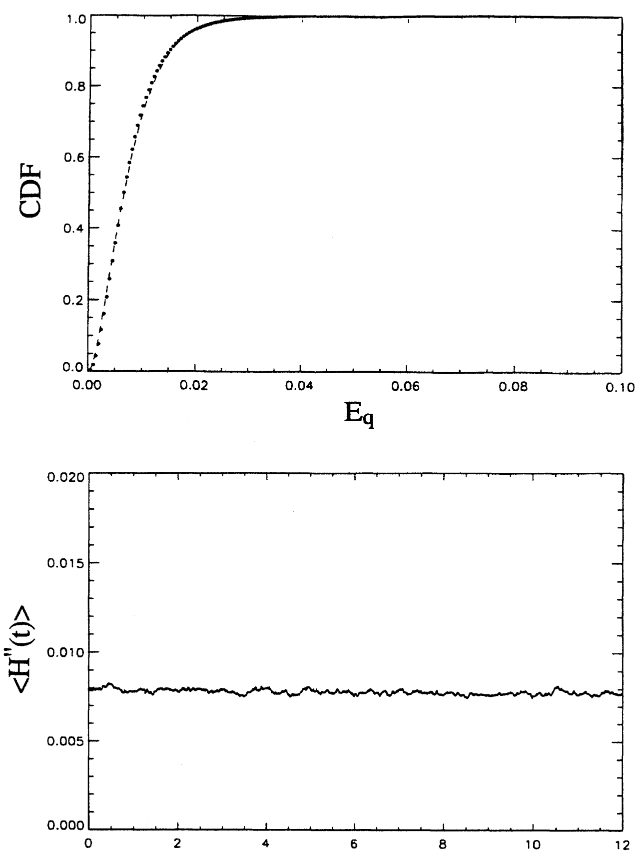

FIG. 2. Same as Fig. 1 with $K_C = 50$, $K_L = 1$, $W = 0.1$, $\varepsilon = 1.305$, $\sigma^F = 35$, and $\sigma^R = 0$.

0.1. For each case, the lattice-averaged energies $\langle H''(t)\rangle$ are measured at the time following each avalanche cluster. These values, which fluctuate with time, are shown at the bottom of each figure. The lattice-averaged energies $\langle H''(t)\rangle$ shown in the figures are then averaged over time to obtain the "temperatures" ε for each simulation, to be used together with (7) in calculating the CDF, the dashed curve in the top panel of each figure. These dashed, theoretical CDF curves are then compared with the experimentally determined CDF measured from the simulations (dots). Agreement between theory and simulation data is good in Fig. 2, less so in Fig. 1. As expected, models that are closer to mean field (Fig. 2) are better represented by Boltzmann statistics. In addition, we find that in mean field the values of ω_i and therefore ε depend simply on K_T, $(J_2)^2$, and W, the exact expression depending on how H' is scaled.

The results obtained here depend only on two conditions: (1) The system executes small fluctuations around a state of fixed internal energy, and (2) that enough noise is present to allow the system to explore its phase space. It can also be shown [14] that a separate condition requires that the rate of forcing be low. Because the line of reasoning does not depend on the massless nature of the slider blocks, we expect that similar results will be observed in massive slider block simulations [1,16] as well. Since the noise amplitude required to generate the Boltzmann distribution decreases as mean field is approached, we also predict that the amplitude of the external noise should be vanishingly small in the mean field limit. The problems of most interest to both earthquake scientists [3,17] and neurobiologists [6] are in real systems with long-range interactions, which are mean field. It is therefore likely that Boltzmann fluctuations will be important in these systems, and that these may be the origin of extended spatial correlations observed in real earthquake fault systems [18,19].

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