Strongly Inhomogeneous Surface Growth on Polymers

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We study nonlinear surface growth driven by spatially localized noise, a model that can be mapped onto directed polymers with random contact interactions. These systems are *asymptotically free* and show nonperturbative *strong-coupling* behavior on large scales in *one* dimension; hence they are possibly the simplest examples with these properties. The strong-coupling regime represents new universality classes of directed growth and of polymer delocalization transitions, which we analyze in detail.

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Growing interfaces are an important example of scale invariance far from equilibrium. Their effective dynamics on large scales of time and distance is governed by a stochastic evolution equation for the height field $h(r,t)$. The Kardar-Parisi-Zhang (KPZ) equation [1]

$$\partial_t h = \frac{\nu}{2}\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta \tag{1}$$

driven by Gaussian white noise with $\overline{\eta(r,t)\eta(r',t')} = \sigma^2\delta^d(r-r')\delta(t-t')$ [2] has become the "standard model" for such growth processes due to its simplicity, its wide range of phenomenological applications, and its notable links with various other problems ranging from the Burgers equation and turbulence to equilibrium systems with quenched disorder [3]. Above one dimension, the KPZ equation shares with these theoretical cousins a notorious difficulty: it has a strong-coupling phase that is not accessible by standard perturbation theory [4]. In contrast, the asymptotic scaling exponents are known exactly in $d=1$, and there is even an exactly solvable lattice model in the KPZ universality class [5].

In this Letter, we study an even simpler model of directed growth, which has a *nonperturbative strong-coupling phase* already in $d=1$. This model is a "dimensionally reduced" version of the KPZ equation with strong spatial inhomogeneities. It has a locally enhanced or reduced rate of mass deposition onto the surface and completely localized Gaussian noise,

$$\overline{\eta(r,t)} = \rho\,\delta^d(r), \quad \overline{\eta(r,t)\eta(r',t')}^c = \sigma^2\delta^d(r)\delta^d(r')\delta(t-t'). \tag{2}$$

Equations (1) and (2) turn out to define a new universality class which we analyze in detail below. Such inhomogeneities are of phenomenological interest; they can be generated, e.g., by a boundary of the system at $r=0$. The resulting growth is deterministic except at the "defect" $r=0$; in that sense, this model interpolates between completely deterministic growth models and the standard noisy KPZ equation.

Of equal importance is the representation of the model as a *directed polymer* in equilibrium, given by the partition function

$$Z(r_f,t_f) = \int \mathcal{D}r \exp\left\{-\frac{1}{\nu}\int_{t_i}^{t_f} dt\left[\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \lambda\eta(r,t)\right]\right\} \tag{3}$$

in terms of the transversal displacement field $r(t)$. The height field of Eq. (1) is then proportional to the "local free energy,"

$$h(r_f,t_f) = \frac{\nu}{\lambda}\log Z(r_f,t_f), \tag{4}$$

where $Z(r_f,t_f)$ denotes the restricted partition sum over all paths ending at a given point (r_f,t_f) [1]. The potential $\eta(r,t)$ models quenched point disorder that is localized along the defect line $r=0$ (in contrast to the standard KPZ system where the point disorder is statistically uniform). This system is closely related to wetting from a random substrate [6,7], which in the simplest $(1+1)$-dimensional case is described by (3) for an interface $r(t)$ subject to the additional constraint $r(t)\geq 0$. Equivalently, Eq. (3) describes a pair of directed chains with relative distance $r(t)$ and random contact interactions. This system has been studied in a perturbation theory similar to the replica approach [8], which, however, does not give any information on the strong-coupling regime. The interactions (2) between the chains arise, e.g., if each chain is a random sequence of molecules of two different kinds which interact attractively if they are of the same kind but repulsively if they are of a different kind.

The approach of this Letter is twofold. We establish the different universality classes by a field-theoretic renormalization of the dynamic equation (1), and we obtain specific information on the strong-coupling regime from studying the polymer transfer matrix numerically in $d=1$. Our main results follow.

(1) The *phase diagram* of the model depends only on the two effective interaction parameters $\lambda_0 = (\sigma^2/\nu^3)^{1/2}\lambda$ and $\hat{\rho} = \nu\lambda_0\rho_0$ with $\rho_0 = \rho/(\nu\sigma^2)^{1/2}$ and is shown schematically in Fig. 1. (a) For $\lambda_0 = \hat{\rho} = 0$, the growth model is linear and the polymer is a free Gaussian chain [9]. (b) A small nonlinearity $(\lambda_0/2)(\nabla h)^2$ in the growth

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rule is a relevant perturbation for $d \leq 1$: It generates a crossover to the strong-coupling regime. The crossover length $\tilde{\xi}$ is given by $\tilde{\xi} \sim (\lambda_0^2)^{-1/2(1-d)}$ for $d < 1$ and has an essential singularity $\tilde{\xi} \sim \exp(\text{const}/\lambda_0^2)$ at the borderline dimension $d = 1$. For $d > 1$, a small nonlinearity is irrelevant; the transition to the strong-coupling phase takes place at nonzero threshold values $\pm\hat{\lambda}_0^c$. In the polymer picture, this perturbation corresponds to an unbiased random defect, i.e., $\hat{\rho} = 0$. Hence this scenario is in agreement with Refs. [7] and [8], but differs from the results of Ref. [6]. (c) As a function of $\hat{\rho}$, the growing interface undergoes a nonequilibrium phase transition that manifests itself, e.g., in the stationary growth rate $\overline{\partial_t h}$ in a system of size L. In the thermodynamic limit $L \to \infty$, the growth rate is independent of $\hat{\rho}$ for $\hat{\rho} \leq \hat{\rho}_c$, but increases with $\hat{\rho}$ for $\hat{\rho} > \hat{\rho}_c$. A similar transition has been found [10] for ordinary KPZ growth with spatially homogeneous noise. In the polymer system, this is a delocalization transition driven by the average defect potential. For $\hat{\rho} > \hat{\rho}_c$, the defect is effectively attractive and confines the polymer to within a finite transversal distance ξ; for $\hat{\rho} < \hat{\rho}_c$, it is effectively repulsive and leaves the polymer in a deconfined state. The critical value $\hat{\rho}_c$ is zero in the weak-coupling regime, but *negative* in the strong-coupling regime (which corresponds to a defect that is on average repulsive).

(2) On the *strong-coupling critical line* $\hat{\rho}_c(\lambda_0^2)$, the system is in a state of asymptotic scale invariance but strongly broken translational invariance. For large disorder, the disorder-averaged stationary polymer density $\overline{p}(r) \equiv \lim_{t-t_i\to\infty, t_f-t\to\infty} \overline{p}(r,t)$ in a system of transversal size L (which is independent of the boundary conditions at t_i and t_f) develops a downward cusp at the defect and tends to a singular limit density $p_c^*(r) \sim |r|^\theta$ with $\theta > 0$ as $\hat{\lambda}^2 \to \infty$ (i.e., $\tilde{\xi} \to 0$); see Fig. 1(a). This can be attributed to anomalous scaling of the local *contact operator* $\Phi(t) \equiv \delta^d(r(t))$,

$$\overline{\langle\Phi\rangle}(L,\hat{\lambda}^2) = \overline{p}(r=0;L,\hat{\lambda}^2) \sim L^{-d}(\tilde{\xi}/L)^\theta \sim L^{-x} \tag{5}$$

as $\tilde{\xi}/L \to 0$. In the growth model, $\Phi(t)$ corresponds to the *response field* $\tilde{h}(r=0,t)$ defined below. The *contact exponent* $x \equiv d + \theta$ is the basic new scaling index of the strong-coupling regime; we obtain $x \approx 1.28$ in $d = 1$. (Contact exponents in thermal systems have been discussed in [11].) The polymer roughness exponent ζ and free energy exponent ω are found to retain their Gaussian values $\zeta = 1/2$ and $\omega = 0$ also in the strong-coupling regime; see below. These are related to the roughness exponent $\chi = \omega/\zeta = 0$ and the dynamic exponent $z = 1/\zeta = 2$ of the growing interface [12].

(3) The *approach to criticality as* $\hat{\rho} \to \hat{\rho}_c$ is also governed by the contact exponent (a) For $\hat{\rho} > \hat{\rho}_c$, the bound state free energy excess (per unit longitudinal length), corresponding to the excess growth rate, and the localization length have the power law singularities $\overline{\partial_t h}(\hat{\rho}) - \overline{\partial_t h}(\hat{\rho}_c) \sim (\hat{\rho} - \hat{\rho}_c)^{\nu_\parallel}$ and $\xi(\hat{\rho}) \sim (\hat{\rho} - \hat{\rho}_c)^{-\nu_\perp}$, respectively, which are determined by the scaling relation [11] $\zeta\nu_\parallel = \nu_\perp = \zeta/(1 - \zeta x)$. For large disorder, the bound state density also develops a minimum $\overline{\langle\Phi\rangle}(\hat{\rho},\hat{\lambda}^2) \sim \xi^{-x}\tilde{\xi}^\theta$ and, as $\tilde{\xi}/\xi \to 0$, tends to a limit $p_b^*(r) \sim |r|^\theta$ which is zero at the defect; see Fig. 1(b). This may seem paradoxical, since it is the defect that localizes the polymer, but can be understood by noting that $\hat{\rho}_c < 0$: Slightly above the transition, the defect line has a few attractive regions but is mostly repulsive. The bound polymer takes advantage of the attractive regions but avoids the rest of the defect. (b) In the deconfined phase [Fig. 1(c)], the scale ξ governs the crossover to the scaling close to an infinitely repulsive defect $p_\infty^*(r) \sim r^2$.

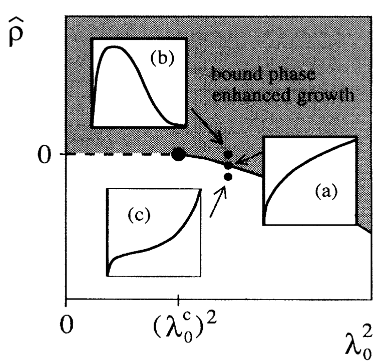

FIG. 1. Phase diagram, parametrized by the effective interaction parameters λ_0^2 and $\hat{\rho}$. The phase of enhanced growth with the bound phase (shaded) terminates on the critical line (thick line) in the weak-coupling regime (dashed) or in the strong-coupling regime (solid). The threshold value $(\lambda_0^c)^2$ is zero for $d \leq 1$. Limit density profiles for $\tilde{\xi} \to 0$ [shown only on one side of the defect (thick margin)]: (a) critical line (p_c^*), (b) bound state (p_b^*), (c) deconfined state (p_∞^*).

Some of these results can be obtained by a renormalization group analysis of Eq. (1) (details of which will be published elsewhere). It is convenient to use the dynamic action [13]

$$J = \int d^d r\, dt_0 \bigg[\frac{1}{2}\delta^d(r)\tilde{h}_0^2 + i\tilde{h}_0 \bigg(\partial_{t_0} h_0 - \frac{1}{2}\nabla^2 h_0 - \frac{\lambda_0}{2}(\nabla h_0)^2 - \rho_0 \delta^d(r) \bigg) \bigg]. \tag{6}$$

The canonical variables $t_0 = \nu t$ and $h_0 = (\nu/\sigma^2)^{1/2} h$ have dimensions $z_0 = 2$ and $\zeta_0 = d - 1$. The conjugate field $\tilde{h}_0$ generates response functions.

The Gaussian theory ($\lambda_0 = \rho_0 = 0$) has the response propagator [Fig. 2(a)]

$$\begin{aligned} G_0(r_2 - r_1, t_{02} - t_{01}) &\equiv \overline{i\tilde{h}_0(r_1,t_{01})h_0(r_2,t_{02})} \\ &= [4\pi(t_{02} - t_{01})]^{-d/2}\theta(t_{02} - t_{01}) \\ &\quad \times \exp[(r_2 - r_1)^2/(t_{02} - t_{01})] \end{aligned} \tag{7}$$

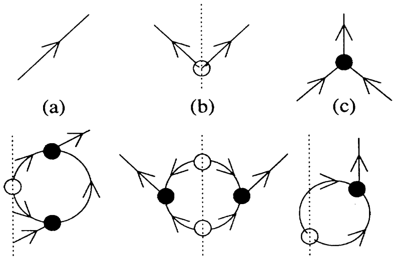

FIG. 2. (a) Response propagator and (b) correlation function of the Gaussian theory. The dashed line marks the defect. (c) The vertex $i\tilde{h}(\nabla h)^2$ (d)–(f) Leading singularities in the diagrammatic series.

and the correlation function [Fig. 2(b)]

$$C_0(r_2, r_1, t_{02} - t_{01}) \equiv \overline{h_0(r_1, t_{01}) h(r_2, t_{02})} = \int dt_0' \, G_0(r_1, t_{01} - t_0') G_0(r_2, t_{02} - t_0') \,. \tag{8}$$

The coupling constant λ_0 has dimension $\varepsilon \equiv 1 - d$. We define the dimensionless coupling $u_0 = \lambda_0 L^{\varepsilon}$; the scale L will also serve to generate the renormalization group flow. In the perturbation series generated by the vertex $i\tilde{h}_0(\nabla h_0)^2$ [Fig. 2(c)], we find for small $\varepsilon > 0$ low-order singularities

$$\overline{i\tilde{h}_0(r_1, t_{01}) h_0(r_2, t_{02})} = \left(1 + \frac{c_1 f(r_1, \varepsilon)}{\varepsilon} u_0^2\right) \times G_0(r_2 - r_1, t_{02} - t_{01}) + O(u_0^4, u_0^2 \varepsilon^0) \,, \tag{9}$$

$$\overline{h_0(r_1, t_{01}) h_0(r_2, t_{02})} = \left(1 + \frac{2c_1 + c_2}{\varepsilon} u_0^2\right) \times C_0(r_1, r_2, t_{02} - t_{01}) + O(u_0^4, u_0^2 \varepsilon^0) \,, \tag{10}$$

$$\overline{h_0(r, t_0)} = \lambda_0 c_3 \int dt_0' G_0(r, t_0 - t_0') + O(u_0^3, u_0 a^0) \,, \tag{11}$$

$c_1 = 2^{-2d-1}\pi^{-d}$ and $c_2 = 2^{-2d-1}\pi^{-d}$ are the residues of the poles of Figs. 2(d) and 2(e). The function $f(r, \varepsilon)$ with $f(r, \varepsilon) \simeq 1$ for $\varepsilon \log(L/|r|) \gg 1$ and $f(r, \varepsilon) \simeq \varepsilon \log(L/|r|)$ for $\varepsilon \log(L/|r|) \ll 1$ constraints the singularity in (9) to the defect line. $c_3 \sim a^{-d}$ is the divergent part of the diagram in Fig. 2(f) in terms of some short-distance cutoff a. All external points in (9), (10), and (11) live on the scale L, i.e., $|r_k| \sim L$ and $t_{0k} \sim L^{z_0}$. The singularities in (9) and (10) can be absorbed into the renormalized fields $h_R = Z h_0$ and $\tilde{h}_R = Z^{-1} \tilde{Z} \tilde{h}_0$ with $Z = 1 - (c_1 + c_2/2) u_0^2/\varepsilon + O(u_0^4)$ and the defect renormalization $\tilde{Z} = 1 - c_1 f u_0^2/\varepsilon + O(u_0^4)$. Since there is no additional renormalization of the vertex $i\tilde{h}_0(\nabla h_0)^2$, these Z factors also determine the coupling constant renormalization [14]:

$$\frac{u_R}{u_0} = \frac{\overline{i\tilde{h}_R(r_1, t_{01}) i\tilde{h}_R(r_2, t_{02}) h_R(r_3, t_{03})}}{\overline{i\tilde{h}_0(r_1, t_{01}) i\tilde{h}_0(r_2, t_{02}) h_0(r_3, t_{03})}} = Z^{-1}. \tag{12}$$

The resulting one-loop beta function $\beta(u_R) \equiv L\partial_L u_R = \varepsilon u_R + (2c_1 + c_2) u_R^3 + O(u_R^5)$ has real-valued fixed points $u_R^{*2} = -\varepsilon/(2c_1 + c_2)$ for $d > 1$, which govern the transition to the strong-coupling regime (cf. Ref. [8]). This transition has the following properties: (i) The canonical time t_0 remains unrenormalized, i.e., $z^* = z_0 = 2$ (and hence $\zeta^* = \zeta_0 = 1/2$). (ii) The height field h_R acquires the scale-dependent dimension $-\chi(u_R) = -\chi_0 + \beta(\partial/\partial u_R)\log Z$ with the fixed point value $-\chi^* = 0$ (and hence $\omega^* = \omega_0 = 0$). (iii) The defect field $\tilde{h}_R(0, t_0)$, which is related to the polymer field $\Phi(t_0)$, acquires the dimension $x(u_R) = 1 + \beta(\partial/\partial u_R)\log[Z^{-1}\tilde{Z}(r = 0)]$ with the fixed point value $x^* = d + 2\varepsilon/3$. (iv) The power singularity (11) signals a shift in $\hat{\rho}_c$. We expect the qualitative properties to be preserved in the strong-coupling phase. However, we find numerically that the contact exponent takes a new value x, which cannot be predicted in perturbation theory.

Our numerical results for the polymer system in $d = 1$ are based on iteration of the lattice transfer matrix derived from (3),

$$Z(r, t + 1) = \exp[\eta(t)\delta_{r,0}] \times \{Z(r, t) + \exp(-1/\nu)[Z(r + 1, t) + Z(r - 1, t)]\} \,, \tag{13}$$

in a system of transversal size L with periodic boundary conditions and $\nu = 1$. The random variables $\eta(t)$ are uncorrelated and take the values $\rho \pm \sigma$ with equal probability 1/2. We hence obtain $\overline{p}(r)$ and the stationary *end-point* density $\overline{p}_f(r) \equiv \lim_{t_f - t_i \to \infty} p(r, t_f)$, namely, $\overline{p}(r) = \overline{Z_1(r, t) Z_2(r, t) / \int Z_1(r, t) Z_2(r, t)\, dr}$ (where Z_1 and Z_2 result from two independent realizations of the randomness) and $\overline{p}_f(r) = \overline{Z(r, t) / \int Z(r, t)\, dr}$ for $t - t_i \gg L^2$. In analogy to (5), $\overline{p}_f$ determines an independent exponent $x_f \equiv d + \theta_f$, the dimension of the *boundary field* $\Phi(t_f)$.

On the strong-coupling critical line, the mean-square displacement from the defect, $\Delta(t_2 - t_1) \equiv \langle \Phi(t_1) r^2(t_2) \rangle$, and the equal-time correlation function of the local free energy, $C(r) \equiv \overline{[h(r, t) - h(0, t)]^2}$, have for $L \to \infty$ the crossover scaling form $\Delta(t; \lambda_0^2) = |t|^{2\zeta_0} \mathcal{D}(|t|^{\zeta_0} \tilde{\xi}^{-1})$ and $C(r; \lambda_0^2) = \log|r/r_0| \mathcal{C}(|r| \tilde{\xi}^{-1})$, respectively. This crossover is shown in Fig. 3; our results indicate that the scaling functions $\mathcal{D}$ and $\mathcal{C}$ have a finite large-distance limit, i.e., $\zeta = \zeta_0 = 1/2$ and $\omega = \chi = 0$ also in the strong-coupling regime.

The crossover scaling of the contact operator Φ is obtained in the four independent ways shown in Fig. 4. (a) The contact probability is of the form

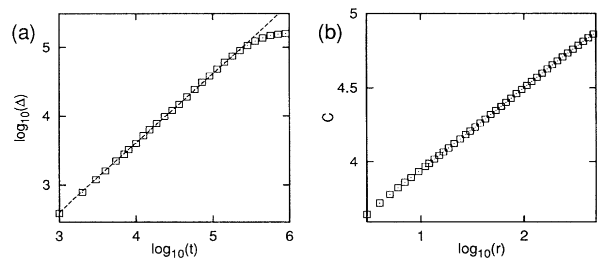

FIG. 3. (a) Mean-square displacement from the defect $\Delta(t) \sim t$ (compared to the weak-coupling case $\sigma = 0$, dashed line) and (b) free energy difference correlation function $C(r) \sim \log|r/r_0|$ on the critical line ($\sigma = 1.832, \rho_c = -1.163$) [compared to $\sigma = 0$, where $C(r) \equiv 0$].

$\overline{\langle \Phi \rangle}(L, \lambda_0^2) \sim L^{-1} \mathcal{F}(L\tilde{\xi}^{-1})$ with the large-scale asymptotics given by (5). (b) The two-point function is expected to factorize, $\overline{\langle \Phi(t_1)\Phi(t_2)\rangle} = R(t_2 - t_1)\overline{\langle \Phi \rangle}$, for $|t_2 - t_1|^\zeta \ll L$, with the L-independent return probability $R(t; \lambda_0^2) = |t|^{-\zeta} \mathcal{R}(|t|^\zeta \tilde{\xi}^{-1}) \sim |t|^{-\zeta x}$ as $|t|^\zeta \tilde{\xi}^{-1} \to \infty$. Off criticality, (c) the localization length ξ, defined, e.g., by $\xi^2 = \int r^2 \overline{p}(r)\, dr$, has the form $\xi(\hat{\rho}, \lambda_0^2) = \hat{\rho}^{-1}\chi(\hat{\rho}\tilde{\xi}) \sim (\hat{\rho} - \hat{\rho}_c)^{-\zeta/(1-\zeta x)}$, and (d) the contact probability is $\overline{\langle \Phi \rangle}(\hat{\rho}, \lambda_0^2) = \hat{\rho}\mathcal{F}_b(\hat{\rho}\tilde{\xi}) \sim (\hat{\rho} - \hat{\rho}_c)^{\zeta x/(1-\zeta x)}$, as $\hat{\rho} \to \hat{\rho}_c$ [15]. All four measurements are consistent with the effective contact exponent $x = 1.28 \pm 0.08$, on the largest numerically accessible scales, but it is difficult to decide whether this is already the asymptotic value. Measurements with analog scaling

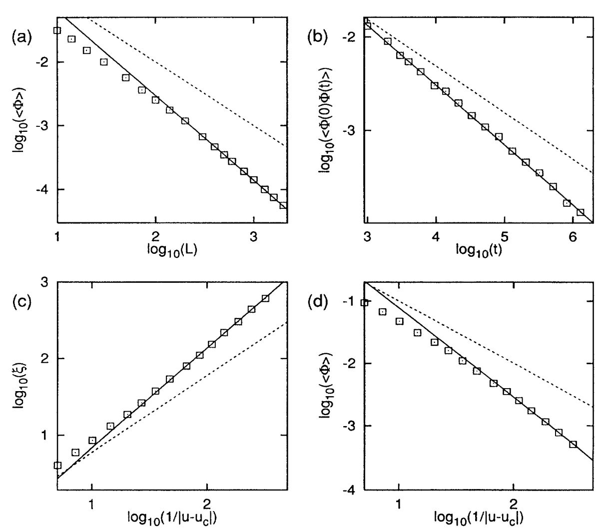

FIG. 4. Determination of the contact exponent x at strong coupling ($\sigma = 1.832$). At criticality ($\rho_c = -1.163$): (a) contact probability $\overline{\langle \Phi \rangle}(L)$; (b) return probability $R(t)$ in a system of size $L = 1000$. Bound phase ($\rho > \rho_c$): (c) localization length $\xi(\rho - \rho_c)$ and (d) contact probability $\overline{\langle \Phi \rangle}(\rho - \rho_c)$; the system size is always $L \gtrsim 4\xi$. All quantities have a consistent power law behavior (solid lines) on large scales that is clearly distinguished from the weak coupling case $\sigma = 0$ (dashed lines).

forms yield $x_f = 1.16 \pm 0.06$ for the end-point exponent. The numerical data for the semi-infinite system with a wall at $r = 0$ are very similar to Figs. 3 and 4, and lead to the same effective exponent. We hence believe the two systems to be in the same universality class.

In summary, we have shown that a model of nonlinear surface growth in one dimension [or directed polymers interacting with a quenched random defect in (1 + 1) dimensions] has a nonperturbative, asymptotically free strong-coupling phase. Hence it is closely related to ordinary KPZ growth in two dimensions [or directed polymers with quenched bulk disorder in (1 + 2) dimensions]. However, this model can be reformulated as a driven lattice gas problem that is likely to be exactly solvable. If the results of this Letter can indeed be established at the level of an exact solution, they are an ideal test for approximate renormalization schemes. Hopefully, they will thus improve our analytic understanding of strong-coupling behavior for systems out of equilibrium.

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[15] The zero of this function also serves to locate the critical value ρ_c numerically. The bound-state probability distribution $\overline{p}(r; \rho, \sigma)$ indeed develops a minimum in the vicinity of the defect for $\xi \gtrsim \tilde{\xi}$.