Exact Soliton Solutions for a Spin Chain with an Easy Plane

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Exact soliton solutions for the Landau-Lifschitz equation for a spin chain with an easy plane are found by using the method of Darboux transformation matrix.

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Some years ago, a 1D magnetic system with an easy plane anisotropy and in an applied magnetic field was mapped onto a sine-Gordon (sG) system [1]. Nevertheless, apart from the questions concerning quantum effects, which are particularly bothersome for the case of $CsNiF_3$ with spins $S = 1$, the mapping from a spin model to a sG system has not been rigorously established except for $T \rightarrow 0$, where, in any case, quantum effects are crucial [2].

As shown by neutron-scattering experiments [3] in $CsNiF_3$, the mapping of an easy plane ferromagnet in a field to a sG system is inadequate. This shows also that out-of-plane fluctuations are of crucial importance at high temperatures. At sufficiently high temperatures, one naturally expects isotropic spin chain dynamics to become a satisfactory approximation [4]. At moderately high temperatures, the effects due to easy plane anisotropy play an important role. The Landau-Lifschitz (L-L) equation for a spin chain with an easy plane was previously unsolved [5]. It is impossible to find, as mentioned by Tjio and Wright [6], the general stationary (i.e., depending on $x - vt$) solution. Solutions of this type given in previous works [7,8] do not satisfy the equation even in the approximation of first order anisotropy. For the same reason, the attempt by means of the direct method of Hirota [9] was unsuccessful. Another attempt [10] was made to reduce the equation to an approximate equation, and a solution was found. But it could not be considered as an approximate solution of the L-L equation with an easy plane, since it does not satisfy this equation in the approximation of first order anisotropy, as we have seen by direct substitution.

No solution to the equation was found by the inverse scattering transform [11,12]. In addition to complexity due to the Riemann surface, required by the double-valued function of the usual spectral parameter, there exist other difficulties, as we shall mention at the end of this Letter.

The L-L equation for a spin chain with complete anisotropy has been studied by the method of the Riemann problem [13,14]. It was reduced to a Riemann boundary value problem on a torus, and was then studied in terms of elliptic functions, but the problem became more complicated. Even though the soliton solutions were found, they are hard to transform to those in the limit of easy plane, as mentioned by Faddeev and Takhtajan [15].

Therefore exact solutions, as well as approximate solutions of the first order anisotropy, of the L-L equation for a spin chain with an easy plane did not appear. In this Letter, the equation is solved by the method of the Darboux transformation matrix [16,17]. By introducing a particular parameter, and constructing Darboux matrices, we are able to show that the Jost solutions can be generated and the Lax equations are satisfied, and then soliton solutions can be obtained. We give an explicit expression of the 1-soliton solution in terms of elementary functions of x and t, as an example. A complete paper will be published separately.

The L-L equation for a spin chain with an easy plane is

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{xx} + \boldsymbol{S} \times J\boldsymbol{S}, \qquad |\boldsymbol{S}| = 1, \tag{1}$$

where the diagonal matrix J

$$J = \mathrm{diag}(0, 0, -16\rho^2) \tag{2}$$

characterizes the easy plane, the 12-plane. Here ρ is a real constant and 16 is introduced for later convenience. The Lax pair of the equation can be obtained from the paper of Sklyanin [18] by setting the present J,

$$L = -i\lambda S_3\sigma_3 - i\mu(S_1\sigma_1 + S_2\sigma_2), \tag{3}$$

$$\begin{aligned} M = {}& i2\mu^2 S_3\sigma_3 + i2\mu\lambda(S_1\sigma_1 + S_2\sigma_2) \\ & - i\mu(S_2S_{3x} - S_3S_{2x})\sigma_1 - i\mu(S_3S_{1x} - S_1S_{3x})\sigma_2 \\ & - i\lambda(S_1S_{2x} - S_2S_{1x})\sigma_3, \end{aligned} \tag{4}$$

where the parameters λ and μ satisfy the relation $\lambda^2 = \mu^2 + 4\rho^2$. If one of them is taken as an independent parameter, the other is a double-valued function of it.

In development of the method of the Darboux transformation matrix, it is reasonable to introduce an auxiliary parameter k such that

$$\lambda = 2\rho\frac{k + k^{-1}}{k - k^{-1}}, \qquad \mu = 2\rho\frac{2}{k - k^{-1}}, \tag{5}$$

The Lax equations are then written as

$$\partial_x F(k) = L(k)F(k), \qquad \partial_t F(k) = M(k)F(k). \tag{6}$$

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We shall drop the arguments x and t henceforth, unless necessary.

Since the 12-plane is the easy plane, the asymptotic spin must lie on it; we thus see that $\boldsymbol{S} = \boldsymbol{S}_0 = (1, 0, 0)$ is the simplest solution of (1). The corresponding Jost solution of (6) may be chosen as

$$F_0(k) = U e^{-i\mu(x-2\lambda t)\sigma_3}, \quad U = \tfrac{1}{2}\{I - i(\sigma_1 + \sigma_2 + \sigma_3)\}. \tag{7}$$

We define the Jost solutions $F_1(k)$ by the Darboux matrices $D_1(k)$ such that

$$F_1(k) = D_1(k) F_0(k), \tag{8}$$

where $D_1(k)$ has poles, as we shall discuss. The properties of $D_1(k)$ and its relation to the solution $\boldsymbol{S}$ of (1) will be determined later [see (25)].

It is obvious that

$$\lambda(-\bar{k}) = \overline{\lambda(k)}, \qquad \mu(-\bar{k}) = -\overline{\mu(k)}, \tag{9}$$

$$L(-\bar{k}) = \sigma_1 \overline{L(k)} \sigma_1, \qquad M(-\bar{k}) = \sigma_1 \overline{M(k)} \sigma_1, \tag{10}$$

$$F_0(-\bar{k}) = -i\sigma_1 \overline{F_0(k)}. \tag{11}$$

Hence we have

$$F_1(-\bar{k}) = -i\sigma_1 \overline{F_1(k)}, \qquad D_1(-\bar{k}) = \sigma_1 \overline{D_1(k)} \sigma_1. \tag{12}$$

Suppose k_1 is a simple pole of $D_1(k)$, $-\bar{k}_1$ is also a pole of $D_1(k)$, as seen from (12). When $D_1(k)$ has only these two simple poles we have

$$D_1(k) = C_1 B_1(k), \tag{13}$$

$$B_1(k) = \left\{1 + \frac{k_1 - \bar{k}_1}{k - k_1} B_1 + \frac{-\bar{k}_1 + k_1}{k + \bar{k}_1} \tilde{B}_1\right\}, \tag{14}$$

where C_1, B_1, and $\tilde{B}_1$ are 2×2 matrices independent of k, and $(k_1 - \bar{k}_1) C_1 B_1$ and $(-\bar{k}_1 + k_1) C_1 \tilde{B}_1$ are residues at k_1 and $-\bar{k}_1$, respectively.

From (3), (4), and (7) we can see that

$$L(k) = -L^\dagger(\bar{k}), \qquad M(k) = -M^\dagger(\bar{k}), \qquad F_0^{-1}(k) = F_0^\dagger(\bar{k}). \tag{15}$$

Hence we have

$$F_1^{-1}(k) = F_1^\dagger(\bar{k}), \qquad D_1^{-1}(k) = D_1^\dagger(\bar{k}) = B_1^\dagger(\bar{k}) C_1^\dagger. \tag{16}$$

From (13) and (16) we have

$$C_1 = \sigma_1 \bar{C}_1 \sigma_1, \qquad \tilde{B}_1 = \sigma_1 \bar{B}_1 \sigma_1. \tag{17}$$

Since $D_1(k) D_1^{-1}(k) = D_1^{-1}(k) D_1(k) = I$, in the limit $k \to k_1$, we have

$$B_1\left\{I - B_1^\dagger + \frac{-k_1 + \bar{k}_1}{2k_1} \tilde{B}_1^\dagger\right\} = 0. \tag{18}$$

This manifests degeneracy of B_1. Hence one can write $B_1 = (\alpha_1\ \beta_1)^T (\gamma_1\ \delta_1)$, and then the expression of $\tilde{B}_1$ according to (17). Substituting then into (18), α_1 and β_1 can be expressed in terms of γ_1 and δ_1, and then

$$B_1 = \frac{1}{\Delta_1}\left\{|k_1|^2(|\gamma_1|^2 + |\delta_1|^2)\begin{pmatrix}\bar{\gamma}_1\\ \bar{\delta}_1\end{pmatrix}(\gamma_1\ \delta_1) - k_1(\bar{k}_1 - k_1)\bar{\gamma}_1\bar{\delta}_1\begin{pmatrix}\delta_1\\ \gamma_1\end{pmatrix}(\gamma_1\ \delta_1)\right\}, \tag{19}$$

where

$$\Delta_1 = |k_1|^2 (|\gamma_1|^2 + |\delta_1|^2)^2 - |k_1 - \bar{k}_1|^2 |\gamma_1|^2 |\delta_1|^2. \tag{20}$$

To determine γ_1 and δ_1, we substitute (8) into (6) and (7) with suitable subscripts, and taking the limit $k \to k_1$ we obtain

$$\partial_x\{C_1 B_1 F_0(k_1)\} = L_1(k_1) C_1 B_1 F_0(k_1), \tag{21}$$

$$\partial_t\{C_1 B_1 F_0(k_1)\} = M_1(k_1) C_1 B_1 F_0(k_1). \tag{22}$$

Owing to the degeneracy of B_1, the second factor of the right-hand sides, $(\gamma_1\ \delta_1) F_0(k_1)$ must appear in the left-hand sides in its original form, and hence it is independent of x and t. We simply obtain

$$(\gamma_1\ \delta_1) = (b_1\ 1) F_0^{-1}(k_1), \tag{23}$$

where b_1 is a constant. Hence the Darboux matrices $D_1(k)$ have been determined, except C_1.

In the limit of $k \to 1$, we have from (3)

$$\lambda(k), \qquad \mu(k) \to 2\rho \frac{1}{k-1} + O(1), \tag{24}$$

and then from (21) we obtain

$$(\boldsymbol{S}_1 \cdot \boldsymbol{\sigma}) = D_1(1) \sigma_1 D_1^\dagger(1). \tag{25}$$

In the limit $k \to -1$, we obtain formulas equivalent to (24) and (25) with different forms.

From (16) we have

$$C_1 C_1^\dagger = I. \tag{26}$$

(17) and (26) show that the matrix C_1 is diagonal and

$$(C_1)_{11} = \overline{(C_1)}_{22}, \qquad |(C_1)_{11}| = 1. \tag{27}$$

Hence one can write

$$C_1 = e^{i(1/2)\omega_1\sigma_3}, \tag{28}$$

where ω_1 is real and characterizes the rotation angle of spin in the 12-plane. It should be mentioned that ω_1 may be dependent on x and t. To determine ω_1, one must examine the Lax equations carefully. Since $e^{i(1/2)\omega_1\sigma_3}$ is a rotation around the 3-axis, it does not affect the value of S_3. Substituting (8) into (6), and taking the limits $k \to \infty$ and $k \to 0$, respectively, we obtain

$$\partial_x\{C_1\} = -i2\rho(S_1)_3\sigma_3\{C_1\}, \tag{29}$$

$$\partial_x\{C_1 B_1(0)\} = i2\rho(S_1)_3\sigma_3\{C_1 B_1(0)\}. \tag{30}$$

Comparison of these two equations gives

$$B_1(0) = C_1^{-2}. \tag{31}$$

From (14) and (19), we obtain

$$C_1 = \Delta_1^{-1/2}\begin{pmatrix} \bar{k}_1|\delta_1|^2 + k_1|\gamma_1|^2 & 0 \\ 0 & \bar{k}_1|\gamma_1|^2 + k_1|\delta_1|^2 \end{pmatrix}. \tag{32}$$

Hence the expression of ω_1 is

$$\frac{1}{2}\,\omega_1 = \arctan\left\{\frac{k_1''}{k_1'}\,\frac{|\gamma_1|^2 - |\delta_1|^2}{|\gamma_1|^2 + |\delta_1|^2}\right\}, \tag{33}$$

where the superscripts $'$ and $''$ denote the real and imaginary parts of a constant, respectively.

We have

$$\gamma_1 = f_1 + i f_1^{-1}, \qquad \delta_1 = f_1 - i f_1^{-1}, \tag{34}$$

where

$$f_1^2 = e^{-\Theta_1} e^{i\Phi_1}, \tag{35}$$

$$\Phi_1 = 2\mu_1' x - 2(\mu_1'\lambda_1' - \mu_1''\lambda_1'')t + \Phi_{10}, \tag{36}$$

$$\Theta_1 = 2\mu_1''(x - V_1 t - x_1), \tag{37}$$

$$V_1 = \lambda_1' + \frac{\mu_1'}{\mu_1''}\lambda_1''. \tag{38}$$

Substituting these formulas into (25), we obtain

$$(S_1)_1 = 1 - 2\frac{4k_1''^2/|1 - k_1^2|^2 + (k_1''^2/k_1'^2)\sin^2\Phi_1}{\cosh^2\Theta_1 + (k_1''^2/k_1'^2)\sin^2\Phi_1}, \tag{39}$$

$$(S_1)_2 = \frac{2(4k_1''^2/|1 - k_1^2|^2)\sinh\Theta_1\cos\Phi_1 - 2(k_1''/k_1')(1 - |k_1|^4)/(|1 - k_1^2|^2)\cosh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + (k_1''^2/k_1'^2)\sin^2\Phi_1}, \tag{40}$$

$$(S_1)_3 = \frac{2[2k_1''(1 - |k_1|^2)/|1 - k_1^2|^2]\cosh\Theta_1\cos\Phi_1 + 2[2k_1''^2(1 + |k_1|^2)/k_1'|1 - k_1^2|^2]\sinh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + (k_1''^2/k_1'^2)\sin^2\Phi_1}. \tag{41}$$

These are the expressions of the 1-soliton solution for a spin chain with an easy plane.

It is reasonable to write them explicitly dependent on ρ. It is convenient to introduce an auxiliary parameter ζ such that

$$\lambda = \zeta + \rho^2\zeta^{-1}, \qquad \mu = \zeta - \rho^2\zeta^{-1}. \tag{42}$$

Comparing with (4) and (5), we have

$$k = \frac{\zeta + \rho}{\zeta - \rho}. \tag{43}$$

It is obvious that $\zeta = \pm\rho$ correspond to zero μ and to $\lambda = \pm 2\rho$. In the complex λ plane, these two points are the edges of cuts.

One can then express the expression of the 1-soliton solution in terms of the parameter ζ. We restrict ζ_1 to the upper half plane of complex ζ, and

$$|\zeta_1| > \rho; \tag{44}$$

then from (43) we find

$$k_1'' = 2\rho\frac{\zeta_1''}{|\zeta_1 - \rho^2|^2}, \qquad k_1' = \epsilon\frac{|\zeta_1|^2 - \rho^2}{|\zeta_1 - \rho|^2}, \tag{45}$$

where $\epsilon = \pm 1$ correspond to $k_1' > 0$ and $k_1' < 0$, respectively. (39)–(41) become

$$(S_1)_1 = 1 - 2\frac{\zeta_1''^2/|\zeta_1|^2 + [4\rho^2\zeta_1''^2/(|\zeta_1|^2 - \rho^2)^2]\sin^2\Phi_1}{\cosh^2\Theta_1 + [4\rho^2\zeta_1''^2/(|\zeta_1|^2 - \rho^2)^2]\sin^2\Phi_1}, \tag{46}$$

$$(S_1)_2 = \frac{2(\zeta_1''^2/|\zeta_1|^2)\sinh\Theta_1\cos\Phi_1 - 2[\zeta_1'\zeta_1''(|\zeta_1|^2 + \rho^2)/|\zeta_1|^2(|\zeta_1|^2 - \rho^2)]\cosh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + [4\rho^2\zeta_1''^2/(|\zeta_1|^2 - \rho^2)^2]\sin^2\Phi_1}, \tag{47}$$

$$(S_1)_3 = \frac{2(\zeta_1'\zeta_1''/|\zeta_1|^2)\cosh\Theta_1\cos\Phi_1 + 2[\zeta_1''^2(|\zeta_1|^2 + \rho^2)/|\zeta_1|^2(|\zeta_1|^2 - \rho^2)]\sinh\Theta_1\sin\Phi_1}{\cosh^2\Theta_1 + [4\rho^2\zeta_1''^2/(|\zeta_1|^2 - \rho^2)^2]\sin^2\Phi_1}, \tag{48}$$

which have never been found by any means tried. These expressions depend essentially on two parameters, namely, the two velocities in (36) and (37), which describe a spin configuration deviating from homogeneous magnetization. The center of inhomogeneity moves with a constant velocity, while the shape of the soliton (the direction of magnetization in its center) also changes with another velocity. They cannot be obviously factorized in the form of separated variables even in moving coordinates. Hence, it is hopeless to solve the L-L equation for a spin chain with an easy plane by means of separating variables. Moreover, these properties remain even in the approximation of order of ρ^2; all attempts tried in this approximation were not successful. It is obvious that when $\rho \to 0$ these three expressions recover those for the isotropic chain.

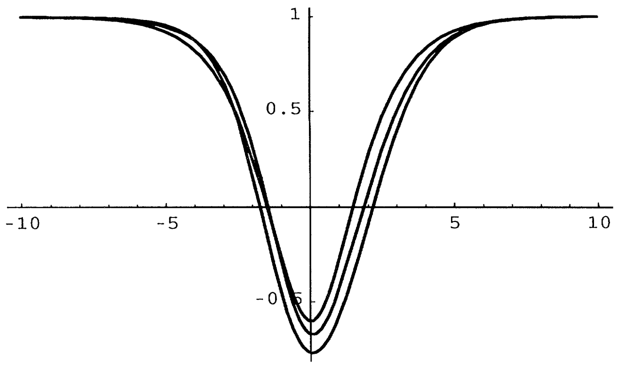

FIG. 1. $\cos\theta(x,t)$: the lowest $t = (0.082)^{-1}(1/4)\pi$, the middle $t = (0.082)^{-1}(1/8)\pi$, the highest $t = 0$.

To show the particular feature of the solution, setting $\rho = 0.112$, $\zeta_1' = 0.1$, and $\zeta_1'' = 0.2$, in the moving coordinates of the soliton, we have

$$\cos\theta = 1 - 2\frac{0.8 + 1.43\sin^2\Phi_1}{\cosh^2\Theta_1 + 1.43\sin^2\Phi_1}, \tag{49}$$

where

$$\Theta_1 = 0.5x, \qquad \Phi_1 = 0.15x + 0.082t. \tag{50}$$

For (a) $t = 0$, (b) $t = (0.082)^{-1}\frac{1}{8}\pi$, and (c) $t = (0.082)^{-1}\frac{1}{4}\pi$, we include Fig. 1.

Noting also that (49) has the property

$$\cos\theta(-x,-t) = \cos\theta(x,t), \tag{51}$$

we see that the depths and widths vary periodically with time and the shapes are not symmetrical with respect to the centers. This feature did not appear in soliton solutions of all other nonlinear equations solved.

On the other hand, $\zeta = \pm\rho$ correspond to $k = \infty$ and $k = 0$. In the above discussion, we have seen that they contribute to the determination of the factor C_1 in (13). This factor is important to ensure that the Jost solution generated satisfies the corresponding Lax equations. This indicates that in the inverse transform the edges of cuts must give a contribution even in the reflectionless case. Unfortunately, Bolovik and Kulinich [12] have apparently not considered these effects. It is natural that they do not obtain any expression for the solution.

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