# Critical Behavior of Superfluid $^4$He in Aerogel

K. Moon
*Department of Physics, University of California, Davis, California 95616*

S. M. Girvin
*Department of Physics, Indiana University, Bloomington, Indiana 47405*
(Received 28 February 1995)

We report Monte Carlo studies of the critical behavior of superfluid $^4$He in the presence of quenched disorder with long-range fractal correlations. Modeling aerogel as an incipient percolating cluster in 3D and weakening the bonds at the fractal sites, *XY*-model simulations demonstrate an increase in the superfluid density exponent $\zeta$ from $0.67 \pm 0.005$ for the pure case to an apparent value of $0.722 \pm 0.005$ in the presence of the fractal disorder, provided that the helium correlation length does not exceed the fractal correlation length.

PACS numbers: 67.40.–w, 61.20.Ja, 64.60.Ak

It is generally believed that the superfluid transition ($\lambda$ point) of pure $^4$He belongs to the classical 3D *XY* model universality class. Near the critical temperature $T_c$, the superfluid density scales as $|T - T_c|^\zeta$, where $\zeta$ is measured [1] to be $0.674 \pm 0.003$. There has been considerable interest in studying the superfluid transition for helium in a variety of porous media [2]. One such system is Vycor which is a random glass with a porosity of order 30%. Remarkably, the critical exponent $\zeta$ is found to be unchanged by the seemingly huge perturbation represented by the glass. This, however, is the result expected according to the Harris criterion [3,4], which shows that weak, uncorrelated randomness is irrelevant at the unperturbed critical point, provided that the specific heat exponent is negative. The exponent [1] $\alpha \approx -0.026 \pm 0.004$ is indeed negative for $^4$He; however, Narayan and Fisher [5] have argued that since $\alpha$ is only slightly negative the crossover to the pure 3D *XY* critical regime is almost logarithmically slow. Until recently Vycor was the only porous medium in which $\zeta$ was observed to be unchanged. New experiments, however, appear to have added a second material, porous gold (which also has short-range correlations), to the list [6].

Aerogel is a fractal silica "dust" with porosities of 95–98% or higher. Despite the fact that aerogel is almost entirely empty space, and that it (unlike Vycor) has only a tiny effect on the critical temperature, the exponent $\zeta$ is apparently quite distinctly shifted [6–8] to approximately 0.75–0.81. The analysis of Ref. [8] includes important corrections to scaling which yield a value of 0.75, which is independent of pressure and hence is presumably the most reliable. This shift of the exponent seems to suggest that the nominally weak aerogel perturbation is relevant at the 3D *XY* critical point and aerogel (perhaps) produces a new universality class. The specific heat data suggest that either hyperscaling is violated or the amplitude ratio is exceptionally small [9,10]. In addition to shifting $\zeta$ and $\alpha$, aerogel dramatically changes the topology of the $^3$He-$^4$He mixture phase diagram [11,12].

It is known that long-range correlations can make disorder relevant, even when $\alpha$ is negative [13,14]. Weinrib and Halperin [13] have demonstrated this for the special case of weak, *Gaussian distributed* disorder with long-range correlations. Li and Teitel [15] have looked at a model of *nonweak* (broadly distributed) but *uncorrelated* disorder and find a nonuniversal increase in the exponent $\zeta$.

Machta [16] has considered a model of aerogel as a relatively uniform medium filled with pores on many different length scales. This model was motivated by early experiments which saw two transitions, one at the usual bulk temperature and one at a slightly lower temperature. Recent improvements in aerogel synthesis techniques appear, however, to have eliminated the larger pores and inhomogeneities so that now only a single transition is observed at a temperature slightly below the bulk value [6]. Huang and Meng [17] have examined a mean-field theory in a percolating cluster system.

We have performed extensive Monte Carlo simulations on the 3D *XY* model for three cases: (i) no disorder, (ii) uncorrelated disorder, and (iii) fractal disorder. We consider lattice sizes up to $24^3$, and use the Wolff algorithm to minimize the otherwise severe effects of critical slowing down [18–20]. The model is defined by compact phase variables $\{\theta\}$ on sites of a simple cubic lattice and has Hamiltonian [21]

$$\frac{H[\theta]}{T} = -\sum_{\mathbf{r},\boldsymbol{\delta}} \frac{K_{\mathbf{r},\boldsymbol{\delta}}}{T} \cos[\theta(\mathbf{r}) - \theta(\mathbf{r} + \boldsymbol{\delta})], \qquad (1)$$

where $\delta$ is a near-neighbor lattice vector. We measured the disorder-averaged superfluid density $\rho_s$ (helicity modulus [21]) using the usual Kubo formula expression [21]. Defining one "sweep" as growing and reorienting a cluster of spins (on the order of the system size when near the critical point) with the Wolff algorithm, typical runs

 0031-9007/95/75(7)/1328(4)$06.00 

involved $5 \times 10^4$ warm-up sweeps, and $2 \times 10^5$ production sweeps, with measurements taken every 200 sweeps. Results were then averaged over typically 100 disorder realizations.

A finite-size scaling analysis is crucial to the accurate determination of the critical temperature and exponents. The scaling ansatz assumes that $\rho_s$ has the form [15,21]

$$\rho_s = TL^{-\Omega}G(L/\xi) = TL^{-\Omega}\tilde{G}[(T - T_c)L^{1/\nu}], \quad (2)$$

where $\Omega = (2 - \alpha)/\nu - 2$. Hence the dimensionless combination $\rho_s L^{\Omega}/T$ is scale invariant at the critical point $T_c$. If we assume that hyperscaling holds, then we have $\Omega = d - 2 = 1$; otherwise, $\Omega$ is *a priori* unknown. In the inset of Fig. 1 we determine the (dimensionless) critical temperature $T_c = 2.156 \pm 0.001$ by plotting $\rho_s L/T$ vs temperature for uncorrelated disorder $K_{\mathbf{r},\delta} = 1 + \delta K_{\mathbf{r},\delta}$ $\delta K_{\mathbf{r},\delta} \in [-\Delta, \Delta]$ and $\Delta < 1$, of relatively large (but bounded) strength $\Delta = 0.7$. Similar calculations for the pure disorder-free ($\Delta = 0$) case yield a distinctly larger value $T_c \cong 2.203 \pm 0.001$. In the main part of Fig. 1, $\{\tilde{G}\}$ is plotted as a function of the scaling variable $(T - T_c)L^{1/\nu}$ with the value $\nu = 0.667 \pm 0.005$ which gives the best data collapse onto a single universal scaling curve. This is the same value of $\nu$ which we obtained for zero disorder and so is consistent with the Harris criterion and the experimental observation in Vycor that uncorrelated randomness does not change the universality class [22]. In order to cross check the above result, we have also measured the magnetization $m$ and computed the Binder ratio [23,24], which is automatically scale invariant at the critical point, $U_4 = 1 - \left[\frac{\langle m^4\rangle}{3\langle m^2\rangle^2}\right]_{\rm ave}$. This allows one to estimate the critical exponent $\nu$ without assuming a value for $\Omega$ as was necessary in the case of the superfluid density. We again found good scaling for the same values of $T_c$ and $\nu$ obtained from $\rho_s$.

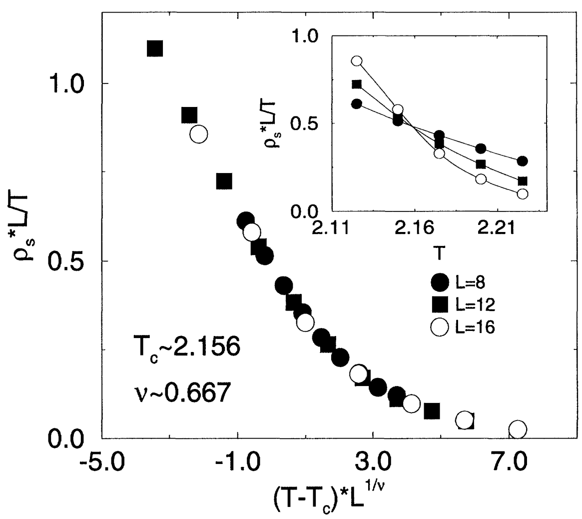


FIG. 1. Universal scaling function $\tilde{G}$ vs the scaling variable $(T - T_c)L^{1/\nu}$ for the case of uncorrelated disorder. The critical exponent $\nu$ is estimated to be $0.667 \pm 0.005$. The inset shows the dimensionless superfluid density $\rho_s L/T$ plotted against temperature.

We turn now to a discussion of fractal disorder. A material with fractal dimension $D_f$ has a mass that scales with length like $M \sim L^{D_f}$. If $D_f < 3$ then (real) objects can never be fractal beyond some finite correlation length $\xi$ because otherwise the density would vanish. Static structure factor measurements [25] indicate that acid-catalyzed aerogel has fractal dimension $D_f \sim 2.4-2.5$ over a wide range of length scales from $\sim 6$ Å out to roughly $\xi \sim 600$ Å. Base-catalyzed aerogel (typically used in the helium experiments) is believed to have a somewhat lower fractal dimension and a lower range of length scales. Various measurements of the "fracton" vibrational properties of aerogels, however, place a *lower bound* on the correlation length for the *connectivity* that is *at least* an order of magnitude larger [26]. The connectivity structure may be important because the closed vortex loops in the helium are presumably attracted to the aerogel strands and are thus sensitive to the connectivity. One of the central mysteries shown up by the experiments is the following. In the critical regime with reduced temperature $t \sim 10^{-4}$ the correlation length of the helium ($\sim 0.3$ $\mu$m) is expected to approach or exceed the (estimated) connectivity length scale. Nevertheless no evidence of a crossover to the uncorrelated disorder regime is evident in the full-pore experiments [2]. Very recently however, Crowell *et al.* [27] have performed experiments in the regime of lower helium densities where even larger correlation lengths can be obtained. They found evidence of a possible crossover to the uncorrelated disorder regime with a lower value of $\zeta$.

We have considered the possibility that the deformability of the tenuous aerogel structure is relevant. On length scales beyond the fracton correlation length, aerogel acts to sound waves like a relatively homogeneous system with a low speed of sound ($\sim 100$ m/s) despite its very low mass density, indicating that its compressibility is nearly $10^6$ times that of ordinary glass. Aerogel is known to be sufficiently flexible that it modifies the collective sound mode dispersion [8,28]. On scales beyond the fracton correlation length, it is reasonable to argue that the aerogel density fluctuations $\delta\Phi$ act as, simple, uncorrelated local *annealed* disorder coupling to the magnitude of the helium order parameter in a Ginzburg-Landau theory with effective Hamiltonian

$$\frac{H}{T} = \frac{1}{2\kappa}(\delta\Phi)^2 + [\tfrac{1}{2} - h\,\delta\Phi]|\nabla\psi|^2 + [\alpha + g\,\delta\Phi]|\psi|^2 + [\lambda + k\,\delta\Phi]|\psi|^4. \quad (3)$$

Integrating out $\delta\Phi$ for small $g, h, k$ produces only irrelevant couplings. For stronger couplings, however, the system can, in the right circumstances, be driven to a tricritical point, beyond which the transition is first order.

This is precisely what happens in $^3$He-$^4$He mixtures where it is a good approximation to treat the $^3$He impurities as annealed disorder [1,27]. This confirms the idea that deformability of the aerogel should be irrelevant. However, it may be more physically correct to include the constraint $\int d^3r\,\delta\Phi = 0$ which would lead to the slow logarithmic case of Fisher renormalization of the critical exponents [29]. The possibility that this may account for the peculiar features of the specific heat data should be looked into in more detail.

We seem to be left only with the possibility of aerogel as quenched disorder whose fractal character extends beyond the lower bound set by the fracton cutoff. A variety of schemes have been used to model the aerogel structure [12,16,17,30]. We have chosen a simple percolation model [30] for the fractal structure (probably more appropriate for acid-catalyzed than base-catalyzed aerogel). We generate a critical percolation backbone on the 3D lattice by randomly occupying lattice sites with probability $p = p_c$, and generate clusters by connecting near-neighbor occupied sites. We then remove all but the largest connected cluster. We selected only fractal realizations with porosity in a narrow window centered on the median value in order to reduce the sample-to-sample fluctuations in the disorder strength. We confirmed that these objects had the known fractal dimension [30] $D_f \sim 2.5$.

In order to be able to do finite-size scaling, while avoiding the scale dependence of the porosity, a single large cluster was generated on an $L_0 = 48$ lattice (giving a porosity of about 95%) and divided into smaller subsystems of size $L = 8, 12, 16, 24$. Simulations were performed for different subsystems with periodic boundary conditions and averaged over the subsystems and different fractal realizations. The $XY$ coupling $K$ was set to unity everywhere except on the ~5% of the bonds which were elements of the fractal where $K$ was arbitrarily reduced to 0.26. The inset of Fig. 2 shows $\rho_s L/T$ vs $T$, and appears to give a clear fixed point with $T_c$ estimated to be $2.183 \pm 0.001$ which is closer to the pure $T_c$ than for the uncorrelated disorder model, since the porosity of the fractal is so high. In the main part of Fig. 2, we plot $\tilde{G}$ vs the scaling variable and find that the (apparent) critical exponent $\zeta$ increases to $0.722 \pm 0.005$. We have confirmed this result with measurements of the Binder ratio. Unlike the case of uncorrelated disorder, we observed a slow drift downward of $U_4$ with system size at the previously determined $T_c$. Taking this out by scaling the data by the factor $U_4(T_c, 0)/U_4(T_c, 1/L)$ yields essentially perfect data collapse with $\nu = 0.722 \pm 0.007$, as shown in Fig. 3. Assuming that a violation of hyperscaling gives an anomalous dimension to the superfluid density $\rho_s \propto L^{-(1+\theta)}\tilde{G}[(T - T_c)L^{1/\nu}]$, we can place an approximate upper bound $|\theta| \leq 0.06$.

It is enlightening to compare the present results to a model with disorder of lower dimension, namely, infinitely long columnar defects. Recent work on this model [31] indicates that $\nu_\perp = \zeta_\perp \sim 1$ is even larger than for the present model. This must be the case in order to satisfy the rigorous lower bound [4] on $\nu$, since this model *does* truly represent a new universality class, and not simply a crossover. The superfluid density measured parallel to the columns has an even larger exponent ($z \equiv \nu_\parallel/\nu_\perp \sim 1.07$).

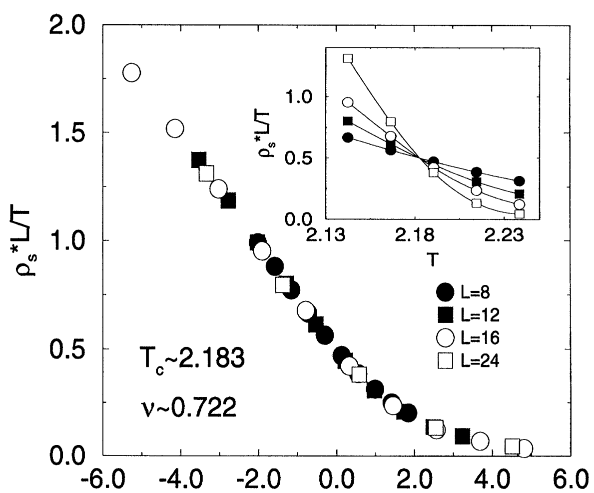


FIG. 2. Universal scaling function $\tilde{G}$ for the case of 95% porosity fractal disorder. The critical exponent $\nu$ is estimated to be $0.722 \pm 0.005$. The inset shows the dimensionless superfluid density $\rho_s L/T$ plotted against temperature.

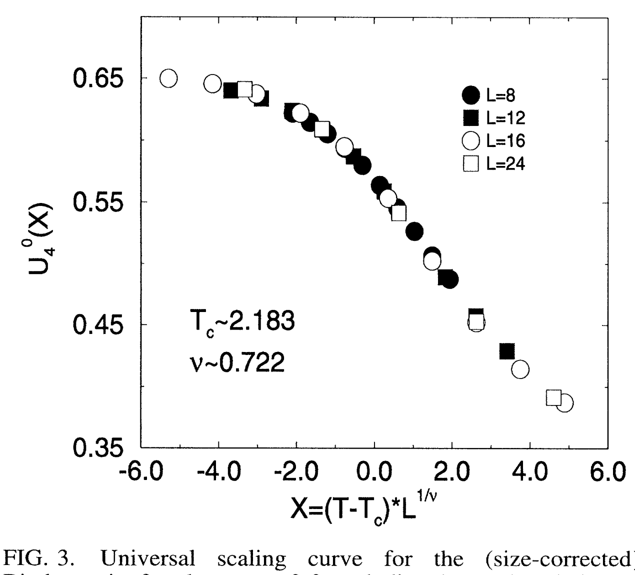


FIG. 3. Universal scaling curve for the (size-corrected) Binder ratio for the case of fractal disorder. The choice of the critical exponent $\nu = 0.722 \pm 0.007$ makes the data for different system sizes collapse onto a single curve and agrees with the conclusion drawn from analysis of the superfluid density.

In conclusion, we have argued that the apparent increase in the superfluid density exponent in aerogel cannot be due to a true change of universality class but must be a crossover effect in the regime where the helium correlation length is less than the (apparently large, but *necessarily finite*) correlation length for the disorder. We have performed Monte Carlo simulations in this regime for a percolation cluster model of fractal disorder and find an increase in the effective exponent to $\zeta = 0.722 \pm 0.005$ which appears to be roughly consistent with experiment. However, we see no apparent violation of hyperscaling, and attempts to confirm the unusual behavior of the experimental specific heat in our model have proved too difficult computationally at this time.

Fractal media are characterized by a mass exponent, a fracton connectivity exponent, and possibly other important exponents. We have presented here a model study of one particular type of fractal. Further work would be useful to investigate the general effect of changing various fractal exponents on the effective values of the helium exponents.

It is a pleasure to thank G. Ahlers, T. Witten, D. S. Fisher, D. Huse, M. W. Chan, C. Hanna, L. Kadanoff, N. Read, J. Machta, M. Cha, G. Zimanyi, R. Scalettar, R. Singh, S. Teitel, J. Reppy, and S. Renn for helpful discussions. The work at Indiana was supported by the NSF through Grant No. DMR-9416906. The work at UC-Davis was supported by NSF Grant No. DMR 92-06023 and by the Los Alamos National Laboratory through a LACOR grant.

[1] G. Ahlers, in *The Physics of Liquid and Solid Helium,* edited by K. H. Bennemann and J. B. Ketterson (Wiley, New York, 1976), Pt. I, Chap. 2.
[2] John D. Reppy, J. Low Temp. Phys. **87**, 205 (1992); M. H. W. Chan, Physica (Amsterdam) **169B**, 135 (1991).
[3] A. B. Harris, J. Phys. C **7**, 1671 (1974).
[4] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).
[5] Onuttom Narayan and Daniel S. Fisher, Phys. Rev. B **42**, 7869 (1990).
[6] Moses Chan (private communication).
[7] M. H. W. Chan, K. I. Blum, S. Q. Murphy, G. K. S. Wong, and J. D. Reppy, Phys. Rev. Lett. **61**, 1950 (1988).
[8] Norbert Mulders, Ravi Mehrotra, Lori S. Goldner, and Guenter Ahlers, Phys. Rev. Lett. **67**, 695 (1991).
[9] G. K. S. Wong, P. A. Crowell, H. A. Cho, and J. D. Reppy, Phys. Rev. Lett. **65**, 2410 (1990); Phys. Rev. B **48**, 3858 (1993).
[10] Melora Larson, Norbert Mulders, and Guenter Ahlers, Phys. Rev. Lett. **68**, 3896 (1992).
[11] S. B. Kim, J. Ma, and M. H. W. Chan, Phys. Rev. Lett. **71**, 2268 (1993).
[12] Recent numerical work has considered $q$-state Potts models [K. Uzelac, A. Hasmy, and R. Jullien, Phys. Rev. Lett. **74**, 422 (1995)], a density function model [L. Pricaupenko and J. Treiner, *ibid.* **74**, 430 (1995)], a density functional model, and the Blume-Emery-Griffiths model [A. Falicov and A. N. Berker, *ibid.* **74**, 426 (1995)] to study the effect of correlated disorder on first-order phase transitions and the phase separation of $^3$He-$^4$He mixtures.
[13] Abel Weinrib and B. I. Halperin, Phys. Rev. B **27**, 413 (1983).
[14] G. Grinstein, in *Fundamental Problems in Statistical Mechanics VI,* edited by E. G. D. Cohen (North-Holland, New York, 1985).
[15] Ying-Hong Li and S. Teitel, Phys. Rev. B **41**, 11 388 (1990).
[16] J. Machta, Phys. Rev. Lett. **66**, 169 (1991).
[17] Kerson Huang and Hsin-Fei Meng, Phys. Rev. B **48**, 6687 (1993).
[18] Ulli Wolff, Phys. Rev. Lett. **62**, 361 (1989).
[19] W. Janke, Phys. Lett. A **148**, 306 (1990).
[20] Cluster algorithms require high quality pseudo-random numbers. See I. Vattulainen, T. Ala-Nissila, and K. Kankaala, Phys. Rev. Lett. **73**, 2513 (1994).
[21] Min-Chul Cha, M. P. A. Fisher, S. M. Girvin, Mats Wallin, and A. Peter Young, Phys. Rev. B **44**, 6883 (1991).
[22] It is possible that there is a logarithmic correction to scaling [5] visible in our data, but it is very weak.
[23] K. Binder, Phys. Rev. Lett. **47**, 693 (1981); Z. Phys. B **43**, 119 (1981).
[24] We thank D. Huse for pointing out that the variance is reduced if one ensemble averages the *ratio* rather than separately averaging the numerator and denominator.
[25] René Vacher, Thierry Woignier, Jacques Pelous, and Eric Courtens, Phys. Rev. B **37**, 6500 (1988).
[26] See, e.g., D. W. Schaefer, C. J. Brinker, D. Richter, B. Farago, and B. Frick, Phys. Rev. Lett. **64**, 2316 (1990), and the many references therein.
[27] P. A. Crowell, J. D. Reppy, S. Mukherjee, J. Ma, M. H. W. Chan, and D. W. Schaefer, Phys. Rev. B **51**, 12 721 (1995).
[28] M. J. McKenna, T. Slawecki, and J. D. Maynard, Phys. Rev. Lett. **66**, 1878 (1991).
[29] M. E. Fisher, Phys. Rev. **176**, 257 (1968).
[30] D. Stauffer, in *On Growth and Form,* edited by H. E. Stanley and N. Ostrowsky (Nijhoff, Dordrecht, 1986), pp. 79–100.
[31] Min-Chul Cha and S. M. Girvin, Phys. Rev. B **49**, 9794 (1994).