

Energy Barriers to Motion of Flux Lines in Random Media

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We propose algorithms for determining both lower and upper bounds for the energy barriers encountered by a flux line in moving through a two-dimensional random potential. Analytical arguments, supported by numerical simulations, suggest that these bounds scale with the length l of the line as $l^{1/3}$ and $l^{1/3}\sqrt{\ln l}$, respectively. This provides the first confirmation of the hypothesis that barriers have the same scaling as the fluctuation in the free energy.

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Glassy states are characterized by a complex energy landscape with many (metastable) free energy minima [1,2]. Some commonly encountered examples are spin glasses, pinned flux lines in superconductors, polymers, domain walls in random field and random bond Ising models, and many more. Typically, the fluctuations between free energy minima in these systems (either in different realizations or in the same random system) scale with observation size L as L^ψ . On the other hand, dynamic response of the system is limited by the barriers in free energy encountered in crossing from one minimum to another. It is likely that the scale of these barriers also grows with size as L^ψ . The simplest assumption is that the two energy scales are comparable, i.e., $\psi = \theta$. However, it is also quite possible that the heights of the ridges in the random energy landscape scale differently from those of the valleys that they separate, with $\psi > \theta$. Yet another scenario is that transport occurs mainly along a percolating channel of exceptionally low energy valleys with $\psi < \theta$. In this paper we examine a specific glassy system, a flux line moving in a two-dimensional random potential, for which we demonstrate $\psi = \theta$.

The system we study is inspired by measurements of nonlinear current-voltage (I, V) characteristics, $V \propto \exp(-\text{const} \times I^{-\mu})$, of disordered superconductors in a magnetic field [3]. There is an emerging consensus that such behavior is best described in terms of low-temperature glass phases [4,5]: In the weak current regime, the dynamical properties are [4,6] dominated by activated processes corresponding to bundles of flux lines overcoming pinning barriers. The principal difference from the classical picture of flux line motion [7] is the power-law growth of barrier energies $E_B \propto L^\psi$, with bundle size L . The latter in turn diverges as the superconducting current driving flux flow decreases, leading to nonlinear (I, V) dependence quoted above. As accurate values of critical exponents characterizing vortex glass phases (including ψ) are presently not available, direct experimental confirmation or refutation of the

various theoretical models is difficult. The point of view taken in many of the pioneering papers [6,8] on the subject is that $\psi = \theta$. Here we shall attempt to put this assumption on a firmer basis.

We shall examine the configurations of a single flux line (FL) in a random potential landscape. Equilibrium properties of this system have been extensively studied in the context of *directed polymers in random media* (DPRM) [9]. It is known that the FL is pinned by impurities into a glassy state. Furthermore, by using a transfer matrix method, properties of this state can be probed numerically in polynomial time in the line length l . In two dimensions, e.g., for a FL trapped between two copper oxide planes of a high- T_c superconductor, some analytical information is also available. For example, the fluctuations in the free energy at finite temperature scale as $l^{1/3}$. Since the scaling behavior of the pinned FL is governed by a zero-temperature fixed point [8], energy fluctuations scale in the same way. The availability of such results and techniques make this system ideal for the investigation of barrier energies.

The precise description of the model is as follows: The FL is discretized to lie on the bonds of a square lattice, directed along its diagonal. Each segment of the line can proceed along one of two directions, leading to a total of 2^l configurations after l steps. These configurations are labeled by the set of integers $\{x(\tau)\}$ for $\tau = 0, 1, \dots, l$, giving the transverse coordinate of the line at each step [clearly constrained such that $x(\tau + 1) = x(\tau) \pm 1$]. To each bond on the lattice a (quenched) random energy is assigned, equally distributed between 0 and 1. The energy of each configuration is the sum of all random bond energies on the line. Without loss of generality, we set $x(0) = 0$. For each end point (l, x) with $x = -l, -l + 2, \dots, l$, there is a configuration of minimal energy $E_{\min}(x|l)$ which can be obtained numerically in a time of order l^2 . It is known that for $|x| < x_c \propto l^{2/3}$ the function $E_{\min}(x|l)$ behaves as a random walk and is thus asymptotically Gaussian distributed [8,9]. We

next examine the energy barrier that has to be overcome when the line is moved from an initial minimal energy configuration between $(0,0)$ and $(t, -x_f)$ to a final one between $(0,0)$ and $(t, +x_f)$.

The only elementary move allowed is flipping a kink along the line from one side to the other (except at the end point). Thus the point (τ, x) can be shifted to $(\tau, x \pm 2)$. Each route from the initial to the final configuration is obtained by a sequence of such elementary moves. For each sequence, there is an intermediate configuration of maximum energy and a barrier that is the difference between this maximum and the initial energy. In a system at temperature T , the probability that the FL chooses a sequence which crosses a barrier of height E_B is proportional to $\exp(-E_B/T)$, multiplied by the number of such sequences. We assume that, as is the case for the equilibrium DPRM, the "entropic" factor of the number of paths does not modify scaling behavior. Thus at sufficiently low temperatures the FL chooses the optimal sequence which has to overcome the least energy, and the overall barrier is the minimum of barrier energies of all sequences.

Since the number of elementary moves scales roughly as the area between the initial and final lines, the number of possible sequences grows as t^{x_t} . This exponential growth makes it practically impossible to find the barrier by examining all possible sequences, hampering a systematic examination of barrier energies. Rather than finding the true barrier energy, we proceed by placing upper and lower bounds on it. The lower bound was given in Ref. [10] and scales as $t^{1/3}$. In this paper, we present an algorithm for obtaining an upper bound. Analytical arguments suggest that this upper bound grows as $t^{1/3}\sqrt{\ln t}$, thus establishing $\psi = 1/3$. Since these arguments do not constitute a rigorous proof, we verify their validity by numerical simulations. Computer time and memory requirements for the construction of this upper bound are happily polynomial in t .

A lower bound for the barrier energy is obtained as follows [10]: Since the end point of the path has to visit all sites (t, x) with $|x| \leq x_f$ and since the energy of any path ending at (t, x) is at least as large as $E_{\min}(x|t)$, the barrier energy cannot be smaller than $\max[E_{\min}(x|t) - E_{\min}(-x_f|t)]$ for $x \in [-x_f, x_f]$. When x_f is sufficiently small, the probability distribution of this lower bound is identical to that of the maximal deviation of a random walk of length x_f . The latter is a Gaussian distribution with a mean value $\propto \sqrt{x_f}$ and a variance $\propto x_f$. This growth saturates for x_f of the order of $t^{2/3}$, leading to the scaling behaviors

$$\begin{aligned} \langle E_-(t, x) \rangle &= t^{1/3} f_1(x/t^{2/3}), \\ \text{var}(E_-) &= t^{2/3} f_2(x/t^{2/3}) \end{aligned} \quad (1)$$

for the lower bound and its variance. The functions $f_1(y)$ and $f_2(y)$ are proportional to \sqrt{y} and y for small y , respectively, and go to a constant for $y = O(1)$. Our simulation results for systems with $t = 256, 512, 1024,$

2048, and 4096 confirm this expectation. Figure 1 shows the scaling functions $f_1(y)$ and $f_2(y)$ for different t , and the collapse is quite satisfactory. However, the initial growth $\propto \sqrt{x_f}$ is not clearly seen at these sizes.

To obtain an upper bound for the barrier, we specify an explicit algorithm for moving the line from its initial to its final configuration. This is achieved by finding a sequence of intermediate steps. It is certainly advantageous to keep the intermediate paths as close to minimal configurations as possible. We first attempt to move the path in steps from a minimal configuration with end point at (t, x) to one with end point at $(t, x + 2)$, starting at $(t, -x_f)$ and ending at (t, x_f) . At each step, we obtain a local barrier path that separates two neighboring minimal paths. The overall barrier is of course the one with the highest energy. While it may occasionally be possible to go from one optimal path to a neighboring one in a single elementary move (as defined above), this is generally not the case. Minimal paths with neighboring end points may be quite far apart at coordinates $\tau < t$. The reason is simple: suppose the random potential has a large positive fluctuation, a "mountain." The minimal energy paths will then circumvent this region by going to its right or left. The last path going to the left and the first one going to the right have almost the same energy (these energies are strictly equal in the continuum limit). They form a *loop* which can be quite large and is likely to enclose the barrier when both paths separate already at small τ . Such loops have been conjectured [4,6] to play an important role in the low-temperature dynamics of the DPRM. Since the transverse fluctuations of a minimal path of length t grow as $t^{2/3}$, we expect the lateral size of these loops to also be of this order.

The algorithm for moving a line of length $t = 2^n$ from a minimal configuration $\{x_1(\tau)\}$ to another one $\{x_2(\tau)\}$,

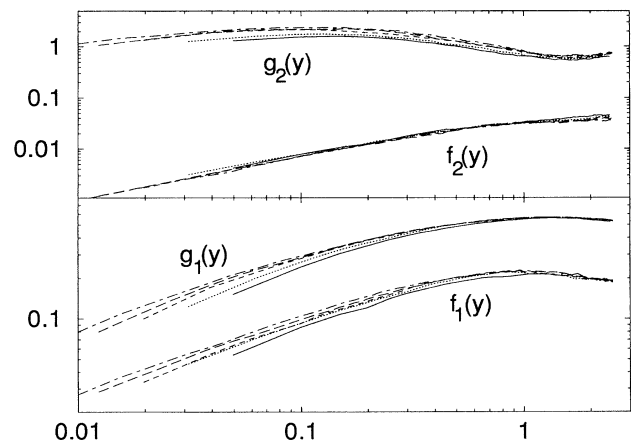


FIG. 1. Scaling functions $f_1(y)$, $f_2(y)$, $g_1(y)$, and $g_2(y)$ [see Eqs. (1), (4), and (5)] for the mean and variance of the lower and upper bounds, averaged over 2000 realizations of randomness, for $t = 256$ (solid), $t = 512$ (dotted), $t = 1024$ (dashed), $t = 2048$ (long dashed), and $t = 4096$ (dot-dashed).

with $x_2(t) = x_1(t) + 2$ is as follows: If $x_2(\tau) \leq x_1(\tau) + 2$ for all τ , we can choose a sequence of elementary moves such that at most two bonds of the line are not on one or the other minimal path, leading to a barrier of order 1 between the two. If $x_2(\tau) > x_1(\tau) + 2$ for some τ , the two paths enclose a loop. We then consider the midway points $(t/2, x)$ with $x_1(t/2) < x < x_2(t/2)$. For each of these points, we find two minimal segments of length $t/2$ connecting on one side to $(0, 0)$ and on the other to either $(t, x_1(t))$ or $(t, x_2(t))$ [11]. The two segments form an almost minimal path of length t , constrained to go through the point $(t/2, x)$. We next move the line $\{x_1(\tau)\}$ stepwise through this sequence of almost minimal paths. At each step we first attempt to move the upper segment and then the lower one [12]. The prescription for moving these segments of length $t/2$ is exactly the same as for paths of length t : If the distance between two consecutive configurations is larger than 2 for some $\tau \in [0, t/2]$, we consider the points at $(t/4, x)$ in between the two, and find minimal paths of length $t/4$ connecting them to the initial and final points. Next we attempt to move segments of length $t/2$ by repeatedly moving line portions of length $t/4$. In some cases, when the energy barrier is high, it is necessary to proceed with this construction until the cutoff scale $t/2^n = 1$ is reached. Thus, at each intermediate configuration, the line is composed of one minimal segment of length $t/2$, one of length $t/4$, etc., ending with two smallest pieces of length $t/2^m$ (equal to 1 in the worst case). The barrier paths resulting from this construction, and the minimal paths separated by them, are shown in Fig. 2.

We now estimate the barrier energy resulting from the above construction. Each intermediate path is composed of segments of minimal paths with constrained end points, and we would like to find the probability distribution for the highest energy. Constraining the end points of a minimal path of length τ typically increases its energy by $E_-(\tau) \propto \tau^{1/3}$. A subset of these intermediate paths (those that cross the largest mountains) have constraints

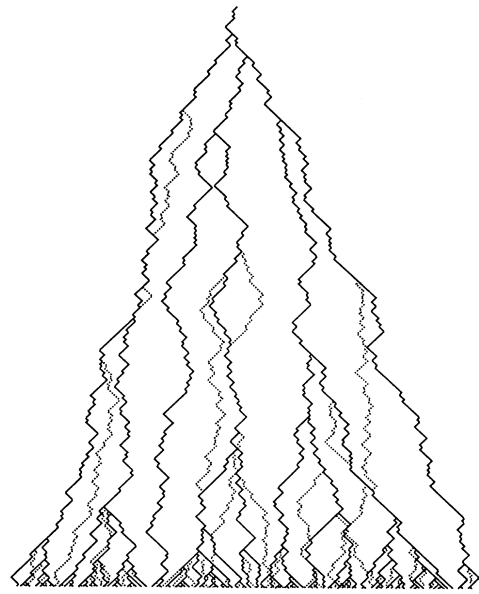


FIG. 2. Minimal paths of length $t = 256$ to end points between $x = -96$ and $x = +96$ (solid) and the barrier paths between them (dotted).

imposed on segments of length $t, t/2, t/4$, and all the way down to unity. The number of paths in this subset (henceforth referred to as *candidate barriers*) grows as $N_c(t) \propto t^\alpha$, with $1 < \alpha < 1 + 2/3$. The lower limit comes from noting that at each bisection of a loop several new large loops are generated, at least one in the upper and one in the lower half of the parent loop, thus $N_c \geq t$. The upper limit comes from the total number of intermediate steps that grows as $t x_f$. The energy of each candidate barrier path is obtained in a manner similar to that of the lower bound: Instead of finding the maximum of a random walk of length $x_f \propto t^{2/3}$, we now have to examine the sum of the maxima for a sequence of shorter and shorter random walks added together. The mean value of this sum is related to the convergent series

$$\begin{aligned} \langle E_c(t) \rangle &= \langle E_-(t) + E_-(t/2) + E_-(t/4) + \dots \rangle = \langle E_-(t) \rangle (1 + 2^{-1/3} + 2^{-2/3} + \dots) + A \\ &\simeq \langle E_-(t) \rangle (1 - 2^{-1/3})^{-1} + A = 4.85 \dots \langle E_-(t) \rangle + A. \end{aligned} \tag{2}$$

The constant A in Eq. (2) accounts for the breakdown of the scaling form of the energy increase for small loops. The mean angle of the smallest loops (of size $t_m = t/2^m$) approaches the 45° limit; their mean energy growing as $0.5 t_m$. For the larger loops, the angle $t_m^{2/3}/t_m$ is small and the energy is $0.23 t_m$. A finite value of m acts as a cutoff separating the two limits. The energy difference per unit length between small and large paths then leads to the additive constant A (of the order of unity) in Eq. (2).

The barrier energy is the maximum of the $N_c(t)$ energies of all candidate barriers. To find its characteristics, we need the whole probability distribution for the en-

ergy $E_c(t)$. Since E_c is the sum of energies coming from its minimal segments, the simplest *assumption* is to regard the segment energies as *independent*, approximately Gaussian, random variables. We then conclude that $E_c(t)$ is also Gaussian distributed with a variance

$$\begin{aligned} \text{var}[E_c(t)] &= \text{var}[E_-(t)] + \text{var}[E_-(t/2)] + \dots \\ &\simeq 2.70 \dots \text{var}[E_-(t)] \propto t^{2/3}. \end{aligned} \tag{3}$$

Since the different segments are in fact constructed through a specific recursive procedure, their independence cannot be justified. Thus the statements of the Gaussian nature of $E_c(t)$ and the variance in Eq. (3) should be

regarded as plausible assumptions that appear to be supported by the numerical simulations.

It can be checked easily that (for large N) the maximum of N independent Gaussian variables of mean a and variance σ^2 is a Gaussian of mean $a + \sigma\sqrt{2 \ln N}$ and variance $\sigma^2/(2 \ln N)$. Since the $N_c(t)$ candidate barriers have large segments in common, their energies are not independent. We can approximately take this into account by assuming a subset of them as independent, leading to $N \propto t^{\alpha'}$ for some $\alpha' < \alpha$. We thus obtain the following estimates for the mean upper bound in barrier energy:

$$\begin{aligned} \langle E_+(x, t) \rangle &= \langle E_c(x, t) \rangle + \sqrt{2 \ln N \operatorname{var} E_c(x, t)} \\ &\simeq (1 + \beta \sqrt{\ln t}) t^{1/3} g_1(x/t^{2/3}), \end{aligned} \quad (4)$$

and its variance,

$$\operatorname{var}[E_+(x, t)] = \frac{\operatorname{var}[E_c(x, t)]}{2 \ln N} \simeq \frac{t^{2/3}}{\ln t} g_2(x/t^{2/3}). \quad (5)$$

The functions $g_1(y)$ and $g_2(y)$ are proportional to \sqrt{y} and y , respectively, for small y , constant for large y , and, in general, different from that of the lower bound.

Our numerical simulations indeed confirm the above scaling forms. The scaling functions $g_1(y)$ and $g_2(y)$ are plotted in Fig. 1 for different values of t , after averaging over 2000 realizations of randomness. The $\sqrt{\ln(t)}$ factors are essential, as a comparable collapse is not obtained without them. In fact the best fit by $\langle E_+(t) \rangle$ is obtained by including the correction to scaling factor $4.85 \langle E_-(t) \rangle$, and with $\beta = 1$. The numerics therefore support the neglect of correlations, and the assumption of a Gaussian distributed $E_c(t)$. As in the lower bound, the initial scaling $\propto \sqrt{x_f}$ is not clearly seen for the sizes studied. Since the leading power for the scaling of the lower and upper bounds is identical, we conclude that the barrier energies also grow as $t^{1/3}$. (It remains to be seen if the logarithmic corrections are truly present, or merely an artifact of our algorithm.)

We now return to the original question of the response of a flux line to an external force, which in the context of superconductivity is proportional to the supercurrent I . The standard argument [4,6] assumes an exponential dependence of the net velocity on the *typical barrier height*. However, it is quite possible that the overall response of the system is determined by the largest rather than the typical barriers. If so, knowledge of the probability distribution of energy barriers is important. For example, it may be more appropriate to average the *waiting time* for the activated jumps over barriers, $\tau \propto \exp(E_B/T)$. Assuming a Gaussian decay in the tail of the barrier heights, the latter average, $[\tau]_{\text{av}} \propto \int dE_B e^{E_B/T} P(E_B)$, is dominated by energies $E_B^* \propto \xi^{\tilde{\psi}}$, with $\tilde{\psi} = 2\psi = 2/3$. More detailed considerations of such issues will be taken up in future publications [13].

In conclusion, for the simple example of a DPRM, we have shown that fluctuations in the minima of the energy landscape and the barriers between them both scale with the length of the line as $t^{1/3}$. It remains to be seen if

the upper bound can be further improved upon and placed on a more firm analytical basis. These results provide a glimpse into the complexity of the free energy landscape of glassy systems.

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Note added.—After submission of the manuscript we learned of similar methods being developed by A. Middleton. One of us (B. D.) succeeded in generalizing our algorithm to a FL in three dimensions, again confirming $\psi = \theta$ (see [14]).

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 - [11] The actual choice depends on the specific orientations of the optimal paths at the two end points. If it is possible to move $(t, x_1(t))$ in the first elementary move, the connection from the midpoints is made to $(t, x_2(t))$. If the move to $(t, x_2(t))$ can be made in the last elementary move, the midway points are connected to $(t, x_1(t))$. If neither is possible, subsequent moves determine this choice.
 - [12] The astute reader may be worried about the feasibility of moving the upper and lower segments independently, given that the midpoint $(t/2, x)$ is common to both. Depending on the orientations of the minimal paths arriving at this point, we found a set of rules on our discrete lattice by which the midpoint could always be shifted when switching from moving one segment to the other. As describing the precise rules is somewhat lengthy, we leave the details to future publications.
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