Long-Range Correlations in Systems with Coherent (Quasi)periodic Oscillations

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Through large-scale simulations of a three-dimensional, deterministic cellular automaton, we demonstrate the existence of algebraic spatial and temporal correlations in nonequilibrium systems that break a continuous time-translation symmetry to produce coherent periodic oscillations. Our results provide the first numerical support for a recent hypothesis that phase fluctuations in such systems are described by the Kardar-Parisi-Zhang equation. The coefficient of the nonlinear term in the equation is determined numerically.

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Rayleigh-Bénard convection, Taylor-Couette flow, and surface waves [1] are among the many extended nonequilibrium systems found experimentally to exhibit phases in which some spatial average, typically a Fourier mode, undergoes regular periodic oscillations in time. Theoretical efforts to understand the spatial coherence necessary to sustain such oscillations (which break spontaneously the time-translation invariance of the underlying equations of motion) have centered primarily on analyzing the dominant low-lying excitations of periodic phases in the presence of external noise. These excitations determine both the conditions under which coherence can be maintained (notably the "lower critical dimension" d_c below which it cannot) and the asymptotic correlations that characterize stable periodic phases for spatial dimensions $d > d_c$.

In a periodic state, the local order parameter $m_t(\mathbf{x})$ has a spatial average $\langle m_t(\mathbf{x})\rangle = f(\omega t)$, where $f(z) = f(z + 2\pi)$ is a periodic function and ω is the oscillating frequency. Local variations can be incorporated by writing $m_t(\mathbf{x}) = f(\omega t + \phi_t(\mathbf{x}))$, where ϕ is a fluctuating "phase" field. Since a uniform ϕ simply shifts the origin of time, leaving the physics unchanged, the dominant "soft" excitations in periodic phases are described by $\phi_t(\mathbf{x})$'s that vary slowly in $\mathbf{x}$ and t. Recently it was argued [2] that in noisy, isotropic systems, such hydrodynamic fluctuations are governed by the Kardar-Parisi-Zhang (KPZ) equation [3]

$$\partial\phi/\partial t = \nu\nabla^2\phi + \frac{\lambda}{2}(\nabla\phi)^2 + \eta(\mathbf{x},t). \qquad (1)$$

Here η is a Gaussian white noise and ν and λ are constants.

The consequences of this prediction are powerful and general: For $d \leq 2$, Eq. (1) is known to have only a "rough" phase wherein the fluctuations, $\langle[\phi(\mathbf{x}) - \overline{\phi}]^2\rangle$, of ϕ diverge algebraically with increasing linear system size L [3], thus destabilizing periodic phases. For $d > 2$, however, Eq. (1) has a "smooth," weak-coupling phase [3] for sufficiently small λ. In this phase, controlled by a fixed point at $\lambda = 0$, the fluctuations are bounded, making stable, globally coherent oscillations possible. Thus d_c is predicted to be 2 [4]. It follows further (see below) that for $d > d_c$ two-point correlations in periodic phases decay asymptotically like $1/r^{d-2}$ or $1/t^{(d-2)/2}$ in space and time, respectively.

Though the chain of arguments underlying these conclusions seems reasonable, it has not been tested experimentally, and has to date received only a very weak numerical test: Consistent with the prediction $d_c = 2$, discrete-time lattice models exhibiting apparently stable quasiperiodic oscillations have been discovered [5] for $d \geq 3$ but thus far not for $d \leq 2$. A detailed characterization of fluctuations in these models, which results from their intrinsic nonlinear dynamics, is lacking. This situation has left room for competing scenarios [6], for the possible existence and origin(s) of oscillating states, which do not invoke the nonlinear diffusive coupling (1).

In this Letter, we present the first confirmation of the predicted algebraic decays of correlations in periodic phases, consistent with the proposition that the dominant hydrodynamic fluctuations are governed by Eq. (1). For a three-dimensional (3D) cellular automaton (CA) model with collective quasiperiodic oscillations, we compute these correlations directly. The behavior of the model with an externally imposed phase gradient confirms explicitly the nonlinear diffusive coupling (1) and yields an estimate for λ. In addition, we examine the role of local correlations in determining the shape of the quasiperiodic orbit in a systematic mean-field approximation. Since the CA we study is noiseless, our calculations also provide evidence that the large-scale behavior of periodic oscillations in systems whose fluctuations are purely internal and deterministic [7] can be the same as that in noisy systems, both being described by Eq. (1).

Due to the efficiency with which they can be implemented numerically, CA furnish a natural testing ground for theoretical predictions about spatial coherence and sta-

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bility in periodic oscillations. The specific 3D CA model we consider was introduced by Hemmingsson [8]. In this model, each site $\mathbf{x}$ of a simple cubic lattice with $N = L^3$ sites is associated with a "spin" variable $\sigma(\mathbf{x}) = 0, 1$. All sites are updated simultaneously according to the rule

$$\sigma_{t+1}(\mathbf{x}) = R_{05}(s_t(\mathbf{x})), \qquad (2)$$

where $s_t(\mathbf{x}) = \sum_{\mathbf{y}\sim\mathbf{x}} \sigma_t(\mathbf{y})$ is a sum over site $\mathbf{x}$ and its six nearest neighbors on the lattice $[0 \le s_t(\mathbf{x}) \le 7]$, and $R_{05}(s) = 1$ for $s = 0, 5$ and $R_{05}(s) = 0$ otherwise. Starting from random initial conditions, the mean variable ("magnetization"), $m_t = N^{-1} \sum_{\mathbf{x}} \sigma_t(\mathbf{x})$, eventually oscillates in time with an irrational period close to 3. Figure 1 shows the return map m_{t+1} against m_t for a 60^3 system with periodic boundary conditions. After a short transient, the orbit converges onto a continuous, closed curve. Successive points on the orbit wind clockwise around the center with a phase velocity $\omega \simeq 0.3371 \times 2\pi$. The scatter of the points away from a single curve is a finite-size effect: Statistics show that the width of the strip containing the orbit decreases as $N^{-1/2}$. For a more complete description of such collective behavior, see [5].

The shape of the orbit (Fig. 1) is seen already for a 10^3 system, suggesting that it is mostly determined by the dynamics of short-distance correlations. To probe this dynamics, we have considered a set of hierarchical equations for the n-point correlations of spins. For example, the one-point function (i.e., mean magnetization) evolves as

$$m_{t+1} = \langle \sigma_{t+1} \rangle = \langle R_{05}(s_t) \rangle, \qquad (3)$$

where $\langle \cdot \rangle$ denotes an average over the initial distribution of spins, which is assumed to be uniform. Writing $R_{05}(s_t)$ as a (unique, seven-variable) polynomial of

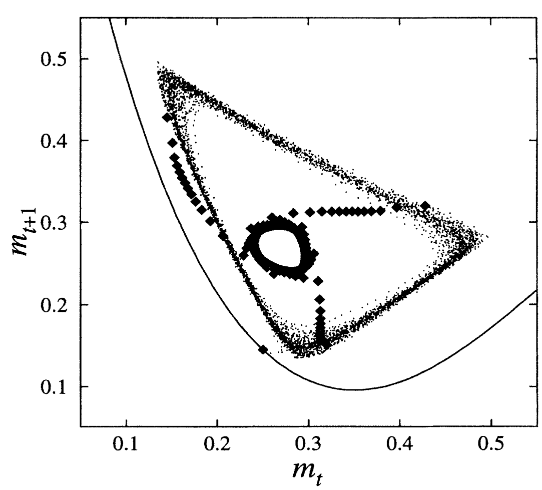

FIG. 1. Return plots of the magnetization $m(t)$ for the Hemmingsson rule on a 60^3 lattice (dots) and the hierarchical approximation truncated at second order (diamonds). The solid line gives the zeroth order mean-field map.

σ_t's, and further expanding around $m_t = \langle \sigma_t \rangle$, we express the right hand side of (3) as a sum of multipoint correlation functions of the variable $\hat{\sigma}_t(\mathbf{x}) \equiv \sigma_t(\mathbf{x}) - m_t$, with coefficients that depend on m_t. The zeroth order term in $\hat{\sigma}_t$ yields the simplest mean-field approximation, $m_{t+1} = (1 - m_t)^7 + 21 m_t^5 (1 - m_t)^2$, which produces the solid line in Fig. 1. Relations analogous to (3) which express multipoint correlation functions of $\hat{\sigma}_{t+1}$ in terms of $\hat{\sigma}_t$ can be obtained in a similar way. They form an infinite coupled hierarchy of iterative equations. Truncation at the second order (keeping only one- and two-point functions) produces the orbit shown by the diamonds in Fig. 1. After an initial transient, this orbit converges to a smooth curve inside the true orbit, with an irrational phase velocity $\omega \simeq 0.2885 \times 2\pi$, close to $4\pi/7$. Within this second-order approximation, correlations beyond a distance of three lattice constants have a negligible effect on the shape of the attractor. Details, higher approximants, and generalizations to other CA will be described elsewhere [9].

Although none of the truncation schemes we have tried predict the actual orbit with great accuracy, they show that low-dimensional maps correctly predict the occurrence of quasiperiodic oscillations in m_t. The phase space of these maps is spanned by the magnetization and a few short-distance correlations, which together approximate the state of the system at a given time. On the other hand, such mean-field schemes typically do not shed light on how long-range spatial coherence is established, and whether it is stable against the intrinsic fluctuations clearly seen in simulations of small systems. Coherence can only be maintained in a large system through the mutual coupling and entrainment of the local phases.

To test the nonlinear diffusive mechanism for such entrainment proposed in Ref. [2] [i.e., the coupling of the phases through Eq. (1)], we have examined the behavior of spin-spin correlations at large distances and long times. On a suitably coarse-grained level, one can write the local magnetization as $m_t(\mathbf{x}) = f(\omega t + \phi_t(\mathbf{x}))$. Fluctuation of $m_t(\mathbf{x})$ away from its spatially averaged value m_t is given by $\delta m_t(\mathbf{x}) \equiv m_t(\mathbf{x}) - m_t \simeq f'(\omega t)\phi_t(\mathbf{x})$ for small $\phi_t(\mathbf{x})$. It then follows that

$$\langle \delta m_t(\mathbf{x}) \delta m_t(\mathbf{y}) \rangle \simeq [f'(\omega t)]^2 \langle \phi_t(\mathbf{x}) \phi_t(\mathbf{y}) \rangle. \qquad (4)$$

In the weak-coupling phase of (1) for $d > d_c = 2$, the right hand side of (4) decays at large distances as $|\mathbf{x} - \mathbf{y}|^{2-d}$. Hence we have

$$\langle \delta m_t(\mathbf{x}) \delta m_t(\mathbf{y}) \rangle \simeq A(t) |\mathbf{x} - \mathbf{y}|^{2-d}, \qquad (5)$$

where the amplitude $A(t)$ is positive. Similar considerations yield the autocorrelation in time,

$$\langle \delta m_t(\mathbf{x}) \delta m_{t'}(\mathbf{x}) \rangle \simeq B(t,t') |t - t'|^{-(d-2)/2}, \qquad (6)$$

where the amplitude $B(t,t')$ oscillates, taking on both signs. Since $m_t(\mathbf{x})$ is a local average of the spins, bare spin-spin correlations have the same large-distance, long-time behavior as those given by Eqs. (5) and (6).

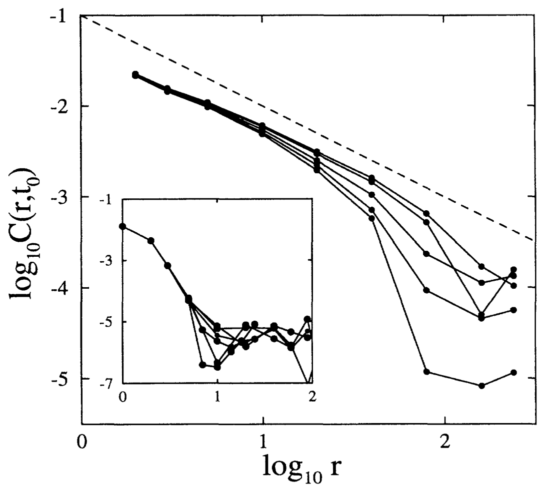

FIG. 2. Log-log plot of the spin-spin correlation function $C(r, t_0)$ for a 512^3 system (periodic boundary conditions). The dashed line has slope -1. From bottom to top: $t_0 = 2048, 2560, 3564, 7680, 15\,872$. Inset: same for rule R_{04}.

Figure 2 shows the two-point spin-spin correlation

$$C(r, t_0) = \frac{1}{NT} \sum_{t=t_0+1}^{t_0+T} \sum_{\mathbf{x}} [\sigma_t(\mathbf{x})\sigma_t(\mathbf{x} + r\mathbf{a}) - m_t^2],$$

obtained from a single simulation of a 512^3 system with an initially random configuration. Here $\mathbf{a}$ is any of the three basis vectors of the lattice, and $T = 89$ is chosen to achieve an approximately uniform sampling of the orbit. Buildup of spatial correlations with increasing time t_0 is clearly seen. For the largest t_0, the data have almost reached saturation (apart from finite-size effects). The correlation is ferromagnetic, and is well described by a $1/r$ decay at large r, consistent with (5).

Figure 3 shows the autospin time-correlation function

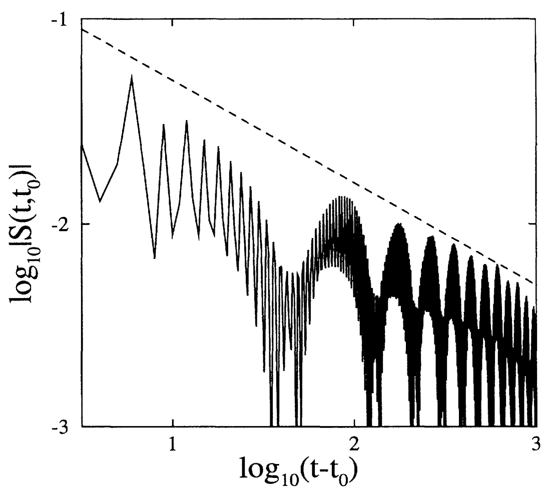

FIG. 3. Log-log plot of the absolute value of the autospin time correlation function $|S(t_0, t)|$ for a 256^3 system with periodic boundary conditions. The dashed line has slope $-1/2$.

$$S(t_0, t) = \frac{1}{N} \sum_{\mathbf{x}} [\sigma_{t_0}(\mathbf{x})\sigma_t(\mathbf{x}) - m_{t_0}m_t],$$

where $t_0 = 5000$. As expected, this function oscillates in t, but the envelope of the oscillation decays as $|t - t_0|^{-1/2}$, consistent with (6).

For comparison, we simulated a different 3D CA model, wherein the function $R_{05}(s)$ is replaced by $R_{04}(s)$, which equals 1 if $s = 0, 4$ and 0 otherwise. This rule produces a state with strong local fluctuations but without global oscillations or other discernible spatial order. Spatial correlations decay much faster (e.g., exponentially) than do those in Hemmingsson's rule. (See the inset of Fig. 2 for the equal-time spin-spin correlation.) The autospin function $S(t, t_0)$ decays much faster with $|t - t_0|$ too. This supports our ascribing the algebraic decays in the original R_{05} CA to the broken symmetry produced by the oscillations.

Returning to the R_{05} rule, since linear diffusion [$\lambda = 0$ in Eq. (1)] would produce identical power laws, we performed a phase-gradient experiment to show that λ is in fact nonzero. We take a system with $N = L_{\parallel}L_{\perp}^2$ sites ($L_{\parallel}$ being the length in the x direction) and periodic boundary conditions. An initially random configuration is iterated 1000 time steps (A system). The lattice is then divided into K slices of width $L_{\parallel}/K$ along the x direction. While keeping the A system running, a new configuration (B system) is constructed by copying consecutive slices from the A system, one at a time, separated by τ iterations. The number τ is chosen such that the orbit points (m_t, m_{t+1}) and $(m_{t+\tau}, m_{t+\tau+1})$ are not too far apart, so that the phase difference between neighboring slices is small. (Typically, we used $\tau = 36$ for $L_{\parallel} = 300$ and $K = 30$.) In this way, an overall phase difference of $\Delta\phi = 2n\pi$ across the B system is established. This new configuration is then iterated further.

Figure 4 shows the evolution of the magnetization profile $m(x, t)$ of a B system with $L_{\parallel} = 300$ and $L_{\perp} = 64$ (x runs horizontally). Each dot represents an average over the spins in a two-dimensional layer perpendicular to the x axis. Amplitude of m sets the gray scale. Successive profiles are separated by 89 time steps. The wave pattern drifts to the left in time at a well-defined velocity $v \simeq 0.021$. Figure 5 shows a plot of v against the phase gradient $u = \Delta\phi/L_{\parallel}$ [10]. The data can be fitted by the functional form $v = (\lambda/2)u + \epsilon u^{-1}$, with $\lambda = 0.6$ and $\epsilon = 0.000\,13$, as shown by the solid line. The u^{-1} term is easily seen to arise from the small residual phase shift, $\delta \equiv 89\epsilon = 89\omega(\mathrm{mod}2\pi) \simeq 0.012$, after 89 iterations in the uniform system; the u term confirms the phase velocity increase, $\Delta\omega = (\lambda/2)u^2$, predicted by Eq. (1) in the presence of a finite phase gradient [11].

To summarize, in the context of a 3D CA model with quasiperiodic oscillations, we have carried out the first test of a general theory of the fluctuations and large-scale behavior of temporally periodic phases that break a continuous time-translation symmetry. Spin-

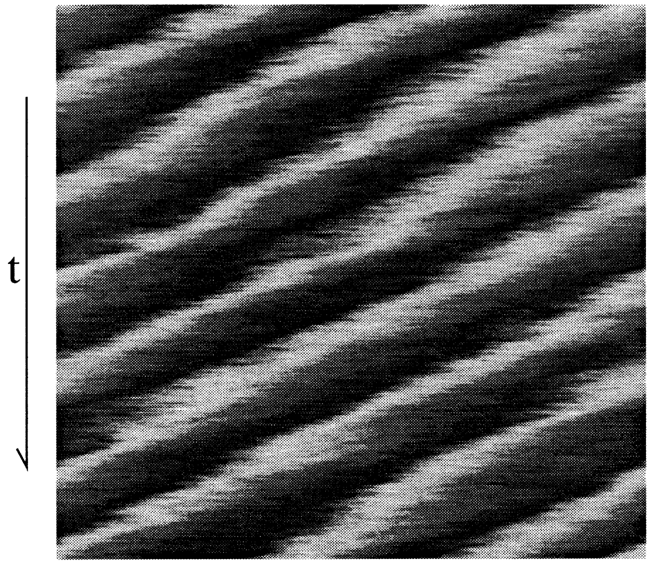

FIG. 4. Time evolution (top to bottom) of layer-averaged magnetization profile (set by gray scale) in a system of size 300×64^2, and a total phase difference $\Delta\phi = 6\pi$ initially distributed over $K = 30$ slices, shown for 300×89 iterations.

spin correlations decay algebraically, with exponents consistent with the hypothesis that fluctuations of the hydrodynamic (phase) variable are governed by the KPZ equation. By studying the dynamics in the presence of a phase gradient, we determined the coefficient of the KPZ nonlinearity. It is interesting that though the theory was designed for noisy systems, it seems to describe correctly the R_{05} CA, whose "noise" is internal and deterministic, thus deterministic chaos and random noise can produce periodic phases with the same asymptotic properties.

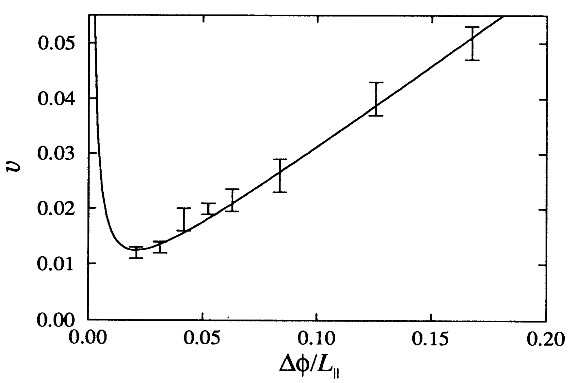

FIG. 5. Equiphase velocity v vs phase gradient $\Delta\phi/L_{\parallel}$ determined graphically from images similar to Fig. 4. The solid line shows a fitted form, consisting of a linear part due to an increased phase velocity, and a correction (pronounced at small gradients) due to the small phase shift of the reference uniform system after 89 iterations.

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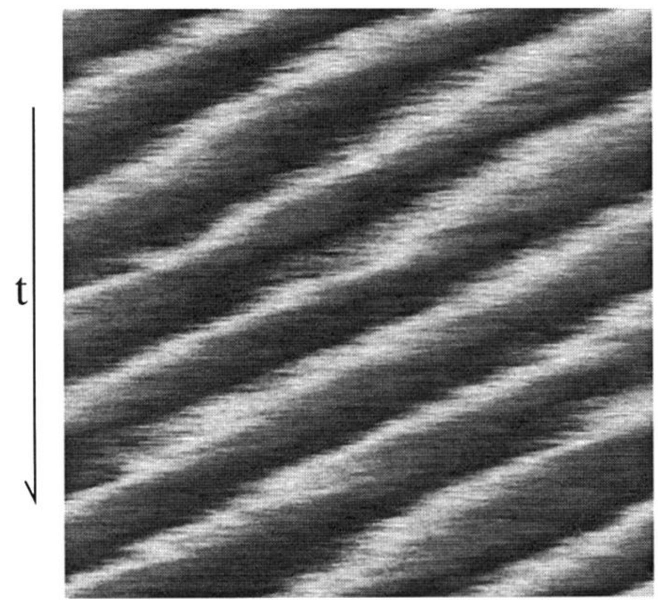

FIG. 4. Time evolution (top to bottom) of layer-averaged magnetization profile (set by gray scale) in a system of size 300×64^2, and a total phase difference $\Delta\phi = 6\pi$ initially distributed over $K = 30$ slices, shown for 300×89 iterations.