Stochastic Noise and Chaotic Transients

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Numerical simulations have been used to determine the influence of stochastic noise on the lifetimes of chaotic transients. The general literature on the subject of noise and chaos appears to suggest that there should be a significant noise dependency, but remarkably *none was found* for the model systems reported here. These null results are confirmed by direct measurements from an electronic analog of one of the systems.

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Deterministic chaos is to some extent the by-product of a system's exquisite sensitivity to initial conditions. When chaos arises, it does so without the requirement of any random ingredient within the system. Yet noise, thermal or otherwise, is always present in any environment. Thus an understanding of chaotic behavior in real-world systems must encompass the interplay between chaos and stochastic forces.

Some of the first studies of the influence of noise on chaotic dynamics dealt with thermally induced escape *from basins of attraction.* For Josephson junctions just prior to a boundary crisis [1,2] and for logistic maps below an interior crisis [3], the mean escape time was found to decrease with increasing noise temperature. Sommerer, Ott, and Grebogi [4] derived a scaling law for the characteristic times in such noise-induced crises. Experimental confirmation of this scaling relationship was provided [5] by measurements of the average time between intermittent chaotic bursts in the vicinity of a crisis for a magnetoelastic ribbon which was driven by an ac magnetic field having both harmonic and random components. Fedchenia *et al.* [6] used both analog electronic and computer simulations to study the influence of noise on periodic attractors in the Lorenz model. They reported evidence of jumps within and between attractors and showed from the Lyapunov exponents that there was noise-induced chaos.

The studies noted above dealt with precrisis systems which would have been nonchaotic in the absence of noise. In contrast, we are interested here in the influence of external noise on a system which is already in a chaotic mode. Such a situation was considered by Franaszek [7] who examined the noise sensitivity of chaotic transients in postcrisis maps. It was reported that the mean transient lifetimes could be extended or shortened, depending on the amount of noise added, and that a particular critical noise amplitude existed for which the mean lifetime of the chaotic transient would be maximized. Recently, Franaszek and Fronzoni [8] experimentally studied noise effects in an analog simulation of a Duffing oscillator biased in a postcrisis condition. In this situation a pair of strange attractors was destroyed, leading to "multitransient" chaos. Again, noise dependence of the lifetimes was seen. However, for the particular system under consideration here, we shall find completely different and somewhat unexpected behavior—thermal noise with no effect on chaotic transients.

Reference [7] was, in fact, the stimulus for the present work, particularly because it drew attention to the use of transient lifetimes as delicate probes of the influence of noise in chaotic systems. Chaotic transients usually arise when a boundary crisis is reached through the adjustment of some control parameter [9,10]. Contact between the boundary of the basin of attraction of a stable attractor and the chaotic strange attractor means that a system will randomly wander over the strange attractor until an abrupt transfer into the basin occurs and chaos ceases. But other scenarios are possible. For example, in a variant of the driven pendulum—with parametric damping instead of the usual velocity dependent dissipation—both chaotic motion [11] and chaotic transients [12] have been seen. In this case the basin of attraction and the strange attractor remnant were spatially congruent, meaning that the transients were not born in a crisis. Very similar chaotic transients are also encountered in a simple pendulum mounted on a vertically oscillating point of suspension [13,14]. The equations of motion of this system are now developed.

Suppose the point of suspension of a simple pendulum—a mass m at a distance r from a pivot—is subjected to a harmonic vertical displacement of amplitude A and frequency ω. The equation of motion of the angular

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coordinate θ is [13,14]

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + mr[g - A\omega^2\cos(\omega t)]\sin(\theta) = \Gamma_0 G(t), \quad (1)$$

where I is the moment of inertia, b is a damping coefficient, g is the acceleration of gravity, and the torque noise on the right hand side is presumed to have the usual time average $\langle G(t)\rangle = 0$ and autocorrelation $\langle G(t_1)G(t_2)\rangle = \delta(t_1 - t_2)$. This white noise term can be approximated using the relationship [15] $G(t) = N(0,1)/\sqrt{\Delta t}$, where $N(0,1)$ is a random variable with Gaussian distribution, zero mean and variance 1, and Δt is a sufficiently fine time grid interval. $N(0,1)$ in turn is easily generated from random numbers uniformly distributed in the interval (0,1) by standard algorithms as described by Knuth [16]. Γ_0 has units of torque per root Hertz, and from the fluctuation-dissipation theorem [15] $\Gamma_0 = (2bkT)^{1/2}$.

A convenient normalization is achieved by employing a new time $\tau = \omega_0 t$ where $\omega_0^2 = g/r$. Then

$$\frac{d^2\theta}{d\tau^2} + \frac{1}{Q}\frac{d\theta}{d\tau} + [1 - \epsilon\Omega^2\cos(\Omega\tau)]\sin(\theta) = [\Gamma_0^*\sqrt{\omega_0}]G^*(\tau), \quad (2)$$

with $\epsilon = A/r$, $Q = \omega_0 I/b$, $\Omega = \omega/\omega_0$, $\Gamma_0^* = \Gamma_0/mgr$, and $G^* = N(0,1)/\sqrt{\delta\tau}$ where $\delta\tau$ is now the normalized time increment. In these units, one cycle of the pendulum's natural oscillations occurs in time 2π, while one cycle of the vertical displacements occurs in time $2\pi/\Omega$. The grid $\delta\tau$ was usually selected to be 1% of the latter period.

Using the fluctuation-dissipation theorem, the right hand side (RHS) of Eq. (2) can be expressed

$$\mathrm{RHS} = \sqrt{\left[\frac{2kT}{I\omega_0^2}\right]\frac{1}{Q\delta\tau}}\,N(0,1) = \sigma N(0,1). \quad (3)$$

The procedure for numerically solving Eq. (2) with given noise prefactor σ consisted of the application of a Runge-Kutta routine on a selected time grid $\delta\tau$. A new random number N was introduced at each new time step [1,3,17].

There are a total of four parameters to be specified: Q, ϵ, Ω, and σ. It was found that the combination $Q = 22$, $\epsilon = 2.03$, and $\Omega = 2.10$ produced chaotic transients of suitable duration. In the noiseless case $\sigma = 0.0$, there are three coexisting stable periodic attractors whose phase plane orbits are illustrated in Fig. 1. One of these is symmetric about the "down" position of the pendulum, while the other two clearly derive from a symmetry-broken "up" orbit. The addition of noise was observed to cause a noticeable blurring of the phase plane trajectories.

Each of the three possible orbits shown in Fig. 1 is completed in two cycles of the harmonic vertical displacement of the pendulum. Therefore, in a Poincaré section, with one sampled point per excitation cycle, any attractor appears as a *pair* of alternately repeating points. When noise is added ($\sigma > 0$), the points become fuzzy. This is illustrated in Fig. 2 for the case $\sigma = 0.04$, which using $\delta\tau = 0.01\ (2\pi/\Omega)$ and Eq. (3) represents an equivalent thermal energy 1.05×10^{-3} times the pendulum energy $\frac{1}{2}I\omega_0^2$.

The origin of the chaotic transient which typically ensues from an arbitrary initial condition (θ, $d\theta/d\tau$) can be understood as follows. There is chaos, so there is a strange attractor. A glimpse of this strange attractor can

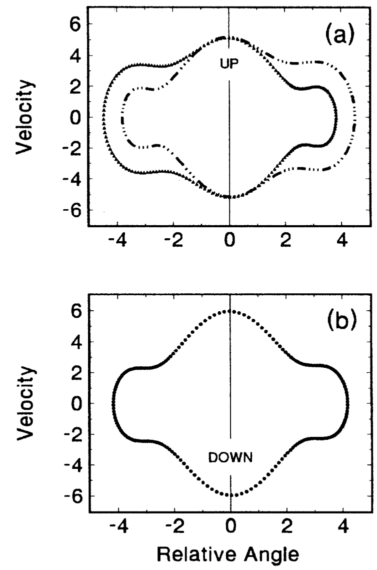

FIG. 1. Phase plane orbits of stable attractors for $Q = 22$, $\epsilon = 2.03$, $\Omega = 2.10$. (a) Pair of broken symmetry orbits referenced to the vertically up position. (b) Symmetric orbit referenced to the vertically down position of the pendulum. Relative angle is measured in radians; velocity is in radians per unit time.

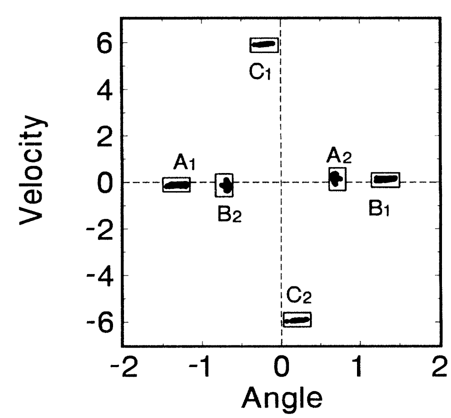

FIG. 2. Poincaré plot of the three stable attractors shown in the previous figure. Each orbit is completed in two cycles of the excitation, so each attractor (A, B, C) contributes two points to this plot, as indicated. The noise amplitude was $\sigma = 0.04$. The boxes are target zones which were used in determining capture by an attractor.

be obtained by increasing ϵ slightly to 2.20 (for which there are no stable orbits and the chaos is permanent); the result is shown in Fig. 3. Also, each of the three periodic states must, of necessity, possess associated basins of attraction. The three separate basins of attraction at $\epsilon = 2.03$ were investigated computationally and found to be composed chiefly of points randomly peppered over the $(\theta,\ d\theta/dt)$ plane, together with some rather faint structural features. This is quite different from the usual appearance of a basin of attraction, which is a closed region in the phase plane with either a simple or fractal bounding curve. But such basins are normally distant from any coexisting strange attractor so that, except at a crisis, there is no contact or overlap. In a manner reminiscent of the parametrically damped pendulum [12], the strange attractor in the present situation has been "destroyed" by the *overlying commingled* basins of attraction for the periodic states; its remnant coexists with these basins.

A chaotic transient ends when the system drops into one of the three available periodic states. Thus the essential task in obtaining data on the duration of transients is the determination of the moment when a point has permanently entered a small target zone surrounding any of the possible fuzzy attractors (Fig. 2). This issue of the cessation of chaos is decided in an *ad hoc* fashion by requiring that 50 successive Poincaré points also fall within the attractor target. Should this condition not be met, the simulation proceeds until a "permanent" capture eventually does occur. Experience has shown that in determining these moments of capture, and hence the lifetimes, it is *critical* that the targets be chosen to be as small as possible yet still large enough to completely surround the extent of the thermal smearing.

For any selected value of σ, Eq. (2) was solved numerically using 25 000 randomly generated initial conditions $(\theta,\ d\theta/d\tau)$ selected from within the phase interval $(-\pi, \pi)$ and velocity interval (-7,7). This particular range in velocity was adopted simply because it was just large enough to encompass the strange attractor (Fig. 3). In each instance the elapsed time before capture by one of the three attractors was determined. The results were then plotted in the form of $\log_e(n_i)$ versus transient lifetime (in units of periods of the excitation), where n_i was the number of transients observed to fall within the range $(200)i \rightarrow 200(i+1)$. The reciprocal slope in such a graph equals the mean transient lifetime [9,12].

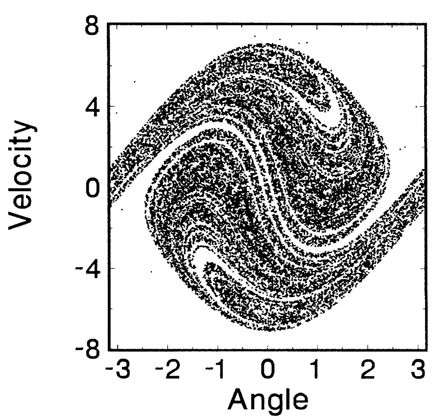

FIG. 3. Chaotic strange attractor at $Q = 22$, $\epsilon = 2.20$, $\Omega = 2.10$. Angle is measured in radians; velocity in radians per unit time.

As can be seen in Fig. 4, the simulations yield essentially indistinguishable data for the three values of σ; the mean lifetime (reciprocal slope) in all cases is 350. The slight apparent scatter at long lifetimes is merely a symptom of the small number of events which occur there. Hence, *the observed mean lifetime seems to be independent of temperature.*

An electronic analog of the vertically oscillating pendulum was constructed according to the design shown in Fig. 5. The inverting input of the operational amplifier is a virtual ground, and summing the node currents

$$C\frac{dV}{dt} + \frac{V}{R_3} + V_s\left[\frac{1}{R_2} + \frac{V_0\sin(\omega t)}{10R_1}\right]\sin(\phi) = \frac{V_n(t)}{R_4}. \tag{4}$$

V is the op-amp output, $V_0\sin(\omega t)$ is an external ac source, the block marked X is a four-quadrant multiplier with internal divide by 10, and the box labeled $\sin(\phi)$ is a VCO subcircuit [18] which generates an output $V_s\sin(\phi)$, where the phase is constrained according to the relation $d\phi/dt = 2\pi KV$. With $\omega_0^2 = 2\pi KV_s/R_2C$ and $\tau = \omega_0 t$, Eq. (5) becomes an exact analog of Eq. (2), provided

$$Q = R_3\sqrt{\frac{C2\pi KV_s}{R_2}}, \quad \epsilon = \frac{V_0R_2}{10R_1\Omega^2}. \tag{5}$$

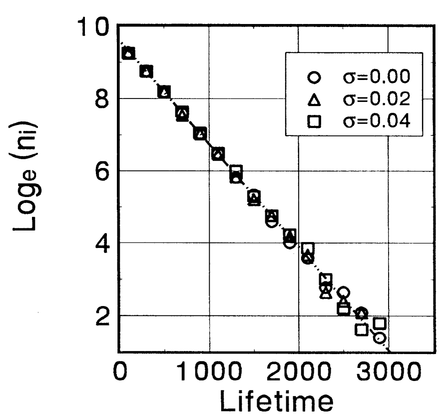

FIG. 4. Results from 25 000 simulation runs to determine lifetimes of chaotic transients with $Q = 22$, $\epsilon = 2.03$, $\Omega = 2.10$. The vertical scale is the natural logarithm of the number of transients found to have the indicated lifetime. The reciprocal slope is a measure of the mean lifetime for the chaotic transients. Data for $\sigma = 0.00$, 0.02, and 0.04 are plotted.

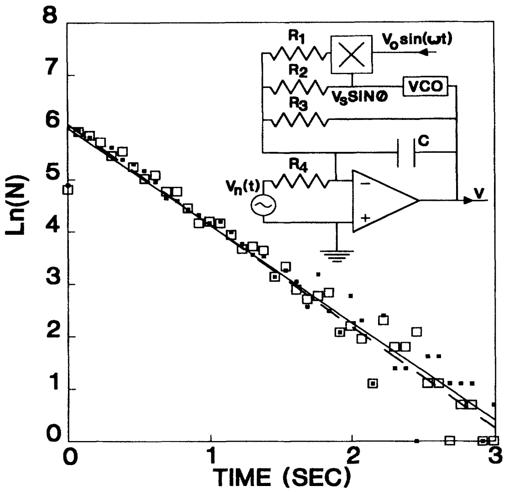

FIG. 5. Measured lifetimes of chaotic transients for zero applied noise (dots) and for $\sigma = 0.05$ (squares). Residual circuit noise contributes approximately $\sigma = 0.0011$. A total of 3000 data points were used in each case. The mean lifetime is estimated to be 0.53 sec. Inset: Schematic diagram of circuit used to electronically simulate the vertically oscillating pendulum.

For convenience the VCO was adjusted so that K was 1000 Hz per volt and $V_s = 1$ V. Other component values were $C = 0.1\ \mu$F, $R_1 = 5\ \text{k}\Omega$, $R_2 = 100\ \text{k}\Omega$, $R_3 = 281\ \text{k}\Omega$, $R_4 = 1317\ \text{k}\Omega$, and hence $\omega_0 = 793\ \text{sec}^{-1}$, and $Q = 22.2$. If $\Omega = 2.10$, then the excitation frequency ω must be set to $1665\ \text{sec}^{-1}$. Then with $V_0 = 4.25$ V, ϵ was 1.92.

The noise generator was a 20 flip-flop shift register circuit with feedback [19]; it was driven at a clock frequency of 100 kHz. The length of a repeat sequence was 10.48 sec—much longer than any of the transient chaotic sequences. The variance in the output distribution from the generator was 0.7 V, and hence $\sigma = 0.7R_2/R_4 \approx 0.05$. Using a computer based data acquisition system, transient lifetimes were measured for 3000 initial conditions, with and without noise. The results shown in Fig. 5 are in essential agreement with the numerical simulation data. Again, there is no discernible effect of noise on the chaotic transients.

Thus we have found that stochastic noise does not affect the lifetime of the chaotic transients which arise in a pendulum with a vertically oscillating pivot. We note parenthetically that the same behavior was also seen in numerical simulations of a parametrically damped pendulum. These results stand in contrast to existing studies of noise in precrisis and postcrisis situations, all of which indicated significant noise influence. The null effect manifested in the counterexample presented here likely has its origin in the specific type of superposition that exists for the attractor basins and the strange attractor remnant.

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