A Choatic Pulsating Star: The Case of R Scuti

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Observational data of the light curve of the variable star R Scuti are subjected to a powerful recently developed nonlinear dynamics approach. This is the first time it is shown that the irregular pulsations of a star are described by a chaotic dynamics with an embedding dimension of 4. The results are relatively robust with respect to the presmoothing of the observational data, as well as to the parameters of the method (such as the delay and sampling rate). This low dimension suggests that the complex pulsational behavior is a result of the nonlinear interactions between just two vibrational modes.

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Irregular stellar variability has fascinated observers for a long time, but theorists have devoted very little attention to this phenomenon, perhaps for want of a suitable mechanism. A recent numerical hydrodynamical survey [1] of the pulsations of cepheid models of the W Virginis type [2] has demonstrated that in a large domain of masses, luminosities, and effective temperatures the computed pulsations are irregular. This behavior is intrinsic to the stellar model and has its origin in the existence of a chaotic dynamics as evidenced by cascades of period doublings that precede the chaotic behavior as one of the model parameters is varied (for a review, cf. [3]). The chaotic dynamics is triggered by a half-integer (5:2) resonance of the excited fundamental mode with the stable second overtone which explains why the behavior is so robust. Calculations show that the model pulsations become increasingly irregular with increasing luminosity and bear a strong resemblance to the higher luminosity cepheids (of RV Tauri type such as R Sct).

However, the question whether the *observed* irregular pulsations of these cepheids are also governed by a chaotic dynamics, and what the dimension of such a dynamics would be, has remained unanswered. In fact, the searches for chaos in observational astronomical data have been unsuccessful, in contrast to other disciplines [4]. The reasons for the general frustration can be found in the large observational errors, but also in the natural and scheduling constraints that make the data acquisition process less systematic than is desirable for most of the sophisticated nonlinear dynamics techniques [5].

A "holy grail" of the nonlinear science community has been the development of techniques to detect the existence and find the characteristics of an intrinsic low-dimensional nonlinear deterministic structure (whether multiperiodic or chaotic), especially in the presence of contaminating noise (for a review, cf. [5,6]). The determination of the dimension of such a dynamics is of great theoretical interest because it is related to the number of variables with which the nonlinear structure can be captured, and the demonstration of a small dimension implies the feasibility of a search for a *simple physical model* of an apparently very complicated phenomenon.

In this Letter we apply a recently developed technique to the analysis of the (RV Tau-type) star R Scuti with the help of the AAVSO observational data [7]. This star displays unsteady pulsations with an approximate "period" of 140 days and with irregularly alternating deep and shallow minima, with intervals of reduced overall amplitude. The observations have been made visually by many amateur observers worldwide. There is considerable scatter in the data as well as observational gaps. The error in the magnitudes is found to have a normal distribution (with a $\sigma = 0.2$), independent of the magnitude. This is perhaps not so astonishing in view of the logarithmic response of the eye, but it forces us to work with the magnitude instead of the more physical luminosity. We follow a standard astronomical preparation procedure [8] that consists of first taking 2.5 day averages of the individual data points followed by a cubic spline smoothing interpolation and a digital low pass filtering. We discuss the effects of variations in the smoothing techniques elsewhere [9,10]. The resultant time sequence, $\{g(t_n)\}$, with a constant, one day separation will be the object of our analysis. Figure 1 shows a typical subsegment of the data set together with the smoothed curve.

Of the whole set of AAVSO observations we have chosen a recent 15 year section which (a) is recent and of denser sampling and (b) is typical, i.e., has stages of both large and small amplitudes that hopefully explore the whole attractor. The smoothed 15 year set is shown in Fig. 2 (top). On the right hand side is the amplitude Fourier spectrum which is quite complicated, with over 30 peaks above the estimated observational noise level. This is significant because on physical grounds the star does not have so many observable $\ell < 3$ eigenfrequencies

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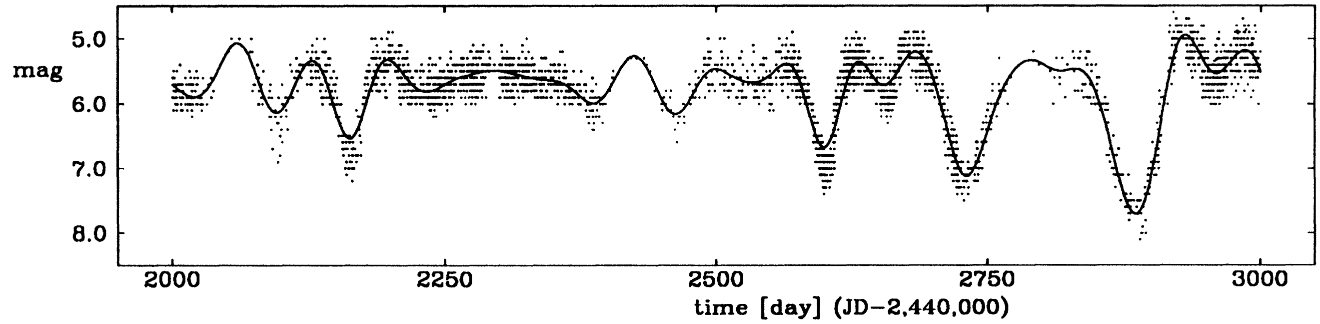

FIG. 1. Magnitude of R Sct vs time; *dots*: individual observations, *solid line*: smoothed data.

(detectable with the whole disk observations). Further, different sections of the data give rise to a different peak structure albeit with a similar envelope [11], a sign that the star is more complicated than multiperiodic, i.e., is probably chaotic. An alternative, viz. a rapid evolution is not supported by the data and is at odds with our current understanding of stellar evolution.

The core of the analysis described in this Letter is a recent global polynomial expansion technique [9,12,13]. Takens' embedding theorem [6] states that if the pulsation is governed by a flow, i.e., by a system of nonlinear differential equations $d\mathbf{Y}/dt = \mathbf{G}(\mathbf{Y})$, where $\mathbf{Y}$ is the d-dimensional vector of the physical phase-space variables, then there exists an embedding variable $\mathbf{X}$ which satisfies a nonlinear equation (map) of the form $\mathbf{X}^{n+1} = \mathbf{F}[\mathbf{X}^n]$; furthermore, an embedding dimension of *at most* $2d + 1$ is required. The set of d_e-dimensional vectors $\mathbf{X}^n = \{g(t_n), g(t_n - \Delta), g(t_n - 2\Delta), \ldots, g(t_n - (d_e - 1)\Delta)\}$ is constructed from the observed scalar variable $\{g(t_n)\} \equiv g(\mathbf{Y}(t_n))$. This global map $\mathbf{F}$ is expanded up to a certain order p with a set of polynomials $P_j(\mathbf{X})$, viz. $\mathbf{F}(\mathbf{X}) = \sum_{k=1}^{p} \mathbf{C}_k P_k(\mathbf{X})$. The polynomials are constructed to be orthonormal on the data set (*natural measure*). Therefore, they are uniquely defined and there are no arbitrary fitting parameters in the approach (besides the time delay Δ).

The procedure consists of constructing such *nonlinear* maps $\mathbf{F}$ as a function of dimension d_e and of determining the minimum d_e which reproduces the observational data with an acceptable accuracy E, e.g., $E = \sum_n |X_i^{n+1} - F_i[\mathbf{X}^n]|^2$ for component $i = 1$. The quantity E compares the "predicted" values to the actual values $\mathbf{X}^{n+1}$. In a noise free situation no improvement should be possible once this minimum embedding dimension has been

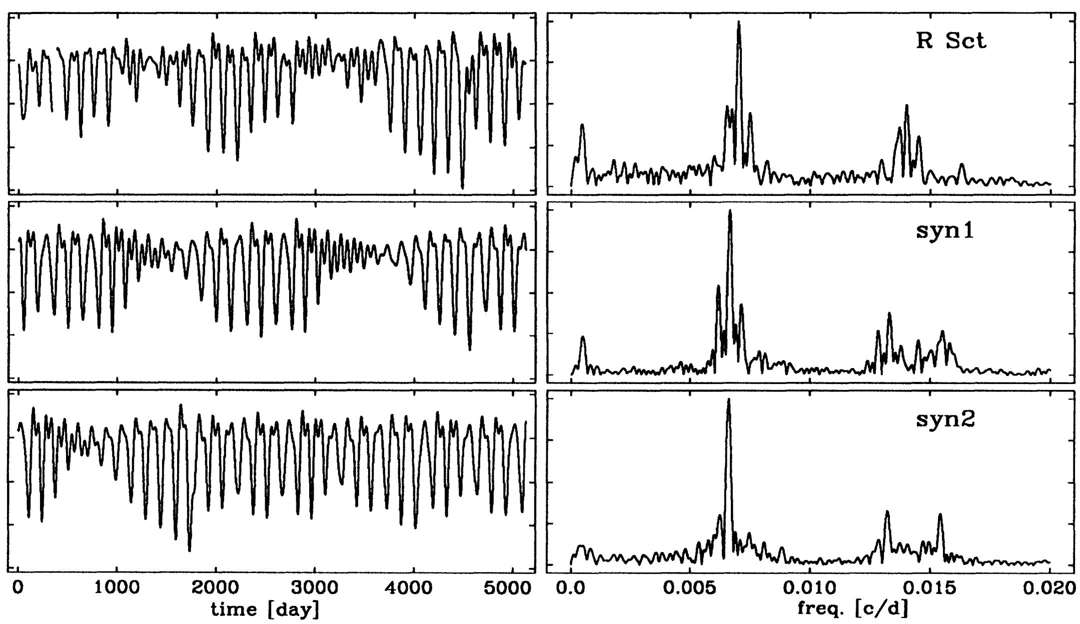

FIG. 2. *Left*: temporal variation of magnitude, *right*: amplitude Fourier spectrum (linear scale), *top*: R Scuti data, *center and bottom*: synthetic data.

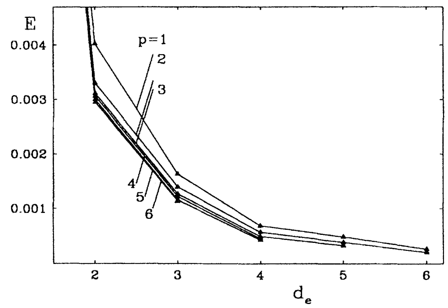

FIG. 3. Error norm E as a function of embedding dimension d_e and order of polynomial p.

reached. According to Takens this d_e thus represents an upper limit on the dimension d of the underlying flow.

Figure 3 shows a clear drop and leveling off suggesting that perhaps $d_e = 4$. This figure suggests that a polynomial expansion of third or fourth order may be sufficient to reproduce the essence of the attractor.

We now show that our 4D and fourth order polynomial map actually captures the stellar dynamics. It is well known that a chaotic map displays an extreme sensitivity to initial conditions, i.e., even the slightest initial difference gets amplified exponentially fast, and a point by point comparison with the data is thus meaningless. Furthermore, the R Sct data are too short to allow the usual statistical tests (e.g., [5]). Instead we have therefore verified that the derived nonlinear map **F** passes the following crucial tests: First, with most arbitrary initial conditions the map generates (synthetic) time sequences that are stable, i.e., they neither blow up, nor do they decay to a periodic cycle. Further, the synthetic signals have a strikingly similar appearance to the R Sct light curve, e.g., the intermittent small amplitude features of the original signal are reproduced, as well as the occurrence of irregularly alternating deep and shallow minima. Two pieces of such a synthetic signal are displayed in column 1 of Fig. 2. Second, and importantly, the synthetic signals have a Fourier spectrum with a statistically similar structure as the R Sct light curve (second column in Fig. 2). Third, after a linear transformation to optimal (Broomhead and King [5]) coordinates the similarity of the light curve and the synthetic signals remains strong. In contrast, the iteration of the second order ($p = 2$) 4D map is unstable, and $p = 3$ yields a limit cycle with a basic frequency of $0.006\,56c/\mathrm{d}$.

We need to address the question of the robustness of the map. The nature of a time sequence generated from even the simplest map, namely the logistic map [6] displays an extreme sensitivity to the parameter of the map because limit cycles and chaotic sequences are intimately mingled. For more complicated maps or flows that lie in a higher-dimensional parameter space the situation is even more complicated. Furthermore, there can be crises points [6] in which a tiny change in the parameters can totally alter the nature of the signal. It should therefore be sufficient, for demonstrating the presence of a chaotic attractor, that the map yields synthetic signals of the same nature as the data *in some neighboring range of parameters*. The polynomial map we have derived necessarily has inaccuracies that are caused by the truncation of the polynomial order p (or possibly by nonpolynomial behavior). To ensure that such inaccuracies do not affect our conclusions we have thus determined the effects of small variations of the map, by multiplying the nonlinear polynomial coefficients by factors of the form $1 + \epsilon$. Returning to the 4D, $p = 3$ map, when we increase ϵ from 0 to 0.2 the limit cycle first turns into a 2-torus, then into chaos with properties similar to those of our 4D, $p = 4$ map. (It is perhaps noteworthy that the two frequencies have a ratio of 2.4 which is reminiscent of the 5:2 resonance which plays a role in onset of chaos in the hydrodynamical models of W Vir models [3].) The fact that the synthetic signal does not *perfectly* match the appearance of the data is therefore not astonishing, and is not even a serious drawback. That the pulsation can be modeled by a 4D map is the important conclusion.

Can we rule out a 3D map? The answer is affirmative: While the synthetic signal, generated with $p = 3$, is chaotic it bears no resemblance to the data nor to the Fourier transform, but is composed of complicated oscillating bursts. With $p = 4$ the latter becomes periodic with the same complicated structure. Finally, with $p = 5$ the synthetic signal turns into a simple almost sinusoidal limit cycle with a frequency of $0.0142c/\mathrm{d}$. [It is worth noting that such a limit cycle is *not* a fixed point of an iterate of the map, i.e., $\mathbf{x} \neq \mathbf{F}^N(\mathbf{x})$]. None of these 3D synthetic signals come close to resembling the data.

Although Δ is a free parameter there are some practical constraints on it which can make it different from the optimal delay one might infer from information theoretical considerations [5,6]. For large Δ the map is very nonlinear and it is necessary to go to a high polynomial order, or worse, the map may not be accurately represented by a polynomial. For small Δ, on the other hand, the nonlinear part of the map may get lost in the noise. We find that for the R Sct data a delay of nine days is a good compromise and the derived low embedding dimension is robust with respect to Δ in a reasonable range of values.

The R Sct data set is too short to lend itself to the calculation of the correlation dimension, but from the iterated map we find a dimension of ≈ 3.1 which confirms the chaotic (strange) nature of the attractor. The Lyapunov exponents are found to be 1.9×10^{-3}, 0.1×10^{-3}, -1.4×10^{-3}, and -5.0×10^{-3}, yielding a Lyapunov dimension [6] of $d_L \approx 3.1$. Both imply that the phase-space dimension $d > 3$. The false neighbor method [14] (kindly performed by Dr. R. Brown) corroborates 4 as the embedding dimension.

The 1D return map generated with our map shows a great deal of structure. It is therefore not astonishing that a 1D return map of the observational data yields an unenlightening scatter diagram [11].

In contrast to a nonlinear map, more standard techniques, for example, a Fourier sum of 30 optimally chosen frequencies (cf. Fig. 2) barely generates a synthetic signal of similar properties. While such a fit can *serve as an interpolation*, when a synthetic signal is generated as a continuation, the appearance is quite different from the R Sct light curve; the Fourier amplitude spectrum is the same, but does not have the proper phase relationships. Standard maps such as the deterministic linear ARMA schemes fall short of producing synthetic signals that are both stable and have the same appearance as the data. On the other hand, it is always possible to cook up stochastic schemes that successfully model the data. However, when our approach was tested on such an apparently similar signal, the polynomial map was unable to produce a decent synthetic signal. Furthermore, it would take a strong *deus ex machina* to cause disturbances of that size in a star. In our opinion such stochastic models appear *ad hoc* compared to a fourth order nonlinear map.

We conclude that the smoothed R Sct data are chaotic and that they are well reproduced by a four-dimensional map. In view of the high noise level in the observational data the important question arises as to whether our results carry over to the actual star. We think they do for the following reasons: First, the main result, namely the dimension, exhibits a certain robustness to the smoothing process. Second, all other explanations appear contrived in comparison to the simplicity of a 4D map. In support we also note that a cursory analysis of another RV Tau-type star, namely AC Her, similarly indicates the presence of a chaotic attractor of the same low dimension, this despite the substantially different appearance of the pulsations.

That the irregular pulsations of a complicated system such as a star should have such a simple underlying dynamics is remarkable. After all, the pulsational behavior involves an intricate interaction between pulsation and heat flow [2]. In addition, models and observations indicate the presence of very strong shock waves and mass loss. From a theoretical point of view our results are thus exciting because they suggest that the irregular pulsations of this star can be modeled by a system of four coupled first order ordinary differential equations. In other words, it may be possible to understand this complicated behavior as arising from the nonlinear coupling of just two vibrational modes. Furthermore, this type of analysis puts constraints on the modeling of these stars which would otherwise not be available. But perhaps the most important potential impact on astronomy is that it may convince observers that useful information can be extracted from the observations of irregular variable stars, and that these stars thus deserve more attention than they have been getting.

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