Violation of Self-Duality for Topological Solitons due to Soliton-Soliton Interaction on a Cylindrical Geometry

R. Dandoloff and S. Villain-Guillot
Groupe de Physique Statistique, Université de Cergy-Pontoise, BP 8428, 95806 Cergy-Pontoise Cedex, France

A. Saxena and A. R. Bishop
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 22 July 1994)

We study classical Heisenberg spins on an infinite elastic cylinder. In the continuum limit the Hamiltonian of the system is given by the nonlinear σ model. We investigate the periodic, cylindrically symmetric solution of the sine-Gordon equation (the Euler-Lagrange equation for this Hamiltonian). The solution does not satisfy the self-dual equations of Bogomol'nyi [Sov. J. Nucl. Phys. **24**, 449 (1976)] which give the minimum energy configuration in each homotopy class. This leads to a novel geometric effect: periodic shrinking of the cylinder.

PACS numbers: 75.10.Hk, 03.65.Sq, 75.80.+q

The interplay of topology and geometry is becoming increasingly prevalent in condensed matter systems as we move to an era of low-dimensional, artificially structured, and nanoscale materials. Here, we consider classical Heisenberg-coupled spins on an infinite elastic cylinder. In the continuum limit, classical one- and two-dimensional static Heisenberg (ferromagnetic and antiferromagnetic) spins are described by a nonlinear σ model [1–4]. First, we consider a rigid cylinder (i.e., a cylinder with constant radius ρ_0); in this case, a solution of the sine-Gordon equation (the Euler-Lagrange equation for this Hamiltonian) is given by a nontrivial spin distribution (soliton). This one-soliton solution satisfies Bogomol'nyi's self-dual equations [5]. The Bogomol'nyi's equations represent a necessary condition to attain the absolute minimum of the energy in each homotopy class. Therefore the one-soliton solution realizes the minimum of the magnetic energy in the first homotopy class.

All solutions of the self-dual equations satisfy the sine-Gordon equation but not vice versa. Indeed, we find that, due to the soliton-soliton interaction, the multisoliton solution of the sine-Gordon equation does not satisfy the self-dual equations, and therefore does not reach the minimum energy (per soliton) in each homotopy class [1]. Thus a variation in the geometry of the cylinder can lead to a lower energy. We find that, for a cylinder with periodic shrinking, the increase of elastic energy is more than compensated by the gain in magnetic energy. Here we note the analogy with the Peierls instability in low-dimensional, interacting electron-phonon systems, although the origin of periodic distortion is quite different. Physically relevant examples include magnetically coated cylindrical thin films or cylinders made from magnetic alloys as discussed below.

The nonlinear σ model for isotropic spin-spin coupling, which is the continuum classical limit of the Heisenberg Hamiltonian for ferromagnets or antiferromagnets [1–4], is given by

$$H = J \int\int_{\text{cylinder}} |\nabla \mathbf{n}|^2 \, dS \,, \tag{1}$$

with $\mathbf{n} = (\cos\theta, \sin\theta\cos\Phi, \sin\theta\sin\Phi)$, and J is the coupling energy between the neighboring spins. We will work in cylindrical coordinates (ρ,x,φ). First, we will adopt homogeneous boundary conditions. Thus we will have different classes of topologically nontrivial spin distributions [i.e., $\theta = \theta(x)$] on the infinite cylinder [1,6].

In this Letter, we restrict ourselves only to solutions with cylindrical symmetry, which will be sufficient for our purposes. This means that θ and Φ will satisfy the following conditions: $\Phi = \varphi$ and $\partial\theta/\partial\varphi = 0$. The Hamiltonian (1) then becomes

$$H = 2\pi\rho_0 J \int_{-\infty}^{+\infty} \left[\left(\frac{d\theta}{dx}\right)^2 + \frac{\sin^2\theta}{\rho_0^2} \right] dx \,. \tag{2}$$

After variation of the Hamiltonian, the Euler-Lagrange equation $\delta H = 0$ leads to

$$\frac{d^2\theta(x)}{dx^2} = \frac{1}{2\rho_0^2} \sin 2\theta \,. \tag{3}$$

The solutions of this sine-Gordon equation (3) are solitons. A solution for a single spin twist is given by the expression

$$\theta = 2 \arctan \exp \frac{x}{\rho_0} \,, \tag{4}$$

which has the following energy relative to the ground state (parallel spins):

$$H_1 = 8\pi J \,.$$

H_1 is the minimum energy level in the first homotopy class and is independent of ρ_0. The solutions corresponding to the absolute minimum energy in each homotopy class also satisfy the self-duality equations

$$\rho_0 \, \partial_x\theta = \pm \sin\theta \, \partial_\varphi\Phi \quad \text{and} \quad \partial_\varphi\theta = \mp \rho_0 \sin\theta \, \partial_x\Phi \,. \tag{5}$$

0031-9007/95/74(5)/813(3)$06.00

For the general case $\theta = \theta(x, \varphi)$ and $\Phi = \Phi(x, \varphi)$, these equations (5) are obtained using the technique employed by Belavin and Polyakov [1], and Bogomol'nyi [5]. Using the obvious expressions

$$\left(\partial_x \theta \pm \frac{\sin\theta}{\rho_0}\partial_\varphi \Phi\right)^2 + \left(\frac{\partial_\varphi \theta}{\rho_0} \mp \sin\theta\,\partial_x \Phi\right)^2 \geq 0\,, \quad (6)$$

we obtain the following inequality:

$$H = J \int \mathcal{H} \;\; \rho_0\, dx\, d\varphi \geq 2J \int \sin\theta\, d\theta\, d\Phi = 8\pi J |Q|\,.$$

The right hand side of the inequality is just the single soliton energy times the winding number of the solution, Q, and does not depend on the geometry of the support. In the case of a cylindrically symmetric solution Eqs. (5) reduce to

$$\rho_0 \partial_x \theta = \pm \sin\theta\,, \quad (7)$$

and the single spin twist soliton $\theta = 2\arctan\exp(x/\rho_0)$ satisfies Eq. (7).

Next, we turn to the periodic solution of the sine-Gordon equation [7]. Note that the Bogomol'nyi argument can be applied on an interval between any two points on the cylinder where θ varies from zero to $n\pi$ and thus covers the sphere S^2 n times. Therefore, the self-dual equations (7) are still valid for this interval and any function that satisfies them would give the minimum energy for this interval. The periodic solution of the sine-Gordon equation (3) can also be obtained directly using a Poisson sum [8]. It is given by the following expression:

$$\theta = \arcsin \operatorname{sn}\left(\frac{x}{k\rho_0}, k\right) + \frac{\pi}{2}\,. \quad (8)$$

The period of this solution is $d = 4\rho_0 k K(k)$, where k is the modulus of the Jacobi elliptic function sn, and $K(k)$ is the complete elliptic integral of the first kind. In the limit $k \to 1$ the period $d \to \infty$ [$\lim_{k\to 1} K(k) \to \infty$] and we recover the single twist soliton [Eq. (4)].

The periodic solution [Eq. (8)] does not reach the minimum energy per soliton H_1. This stems from the fact that the periodic solution [Eq. (8)] *does not* satisfy the self-duality equations (7). The exact magnetic energy per soliton (or the energy density per half period, $d/2$), since $\theta(\pm d/4) = 0[\mathrm{mod}\,\pi]$, reads

$$H = \frac{8\pi J}{k}\left[E(k) - \frac{k'^2 K(k)}{2}\right], \quad (9)$$

where k' is the complementary modulus ($k'^2 = 1 - k^2$) and $E(k)$ is the complete elliptic integral of the second kind. In the dilute limit, i.e., $k \to 1$, $k' \to 0$, and $E(k) \to 1$, we expand the exact solution (9) in powers of k', and find that the energy per soliton is given by

$$H = 8\pi J + J\frac{k'^2}{k\rho_0} = 8\pi J\left[1 + 4\exp\left(-\frac{d}{2\rho_0}\right)\right]. \quad (10)$$

The energy per soliton is higher than the minimum of the energy associated with a single topological soliton: there is an exponentially decaying repulsive interaction energy between two solitons [7,9]. In the single twist soliton limit ($d \to \infty$ and $k' \to 0$) the term $k'^2/k\rho_0$ vanishes and we recover the energy H_1 of the single twist soliton. The periodic soliton solution satifies the relation

$$\rho_0^2(\partial_x\theta)^2 = \sin^2\theta + \frac{k'^2}{k^2}\,. \quad (11)$$

Equation (11) represents a modified "equipartition" relation between "kinetic" $\rho_0(\partial_x\theta)^2$ and "potential" $\sin^2\theta/\rho_0$ energy. For a multisoliton solution, in addition to the potential energy, there is an interaction energy term: $k'^2/\rho_0 k^2$. In the limit of a single twist soliton the interaction term goes to zero and the equipartition holds. The fact that the self-dual equations are not satisfied implies that we can minimize the magnetic energy by an elastic deformation of the cylinder. This possibility arises from the fact that, due to soliton-soliton interaction, the width of the lattice soliton ($k\rho_0$) is smaller than ρ_0. In the case of a single twist soliton, the width of the soliton (4) appears naturally to be ρ_0 (the characteristic length in the problem) [6].

If we relax the constraint $\rho = \rho_0$ and allow $\rho = \rho(x)$, i.e., we allow elastic deformations of the cylinder, the magnetic part of the Hamiltonian (the nonlinear σ model) becomes

$$H_{\mathrm{magn}} = J\int\left[\frac{(\partial_x\theta)^2}{1 + (\partial_x\rho)^2} + \frac{\sin^2\theta}{1 + (\partial_x\rho)^2}(\partial_x\phi)^2 + \frac{(\partial_\Phi\theta)^2}{\rho^2} + \frac{\sin^2\theta}{\rho^2}(\partial_\Phi\phi)^2\right]\rho\sqrt{1 + (\partial_x\rho)^2}\;\, dx\, d\Phi\,, \quad (12)$$

to which we must add an elastic part, which is physically modeled by the following Hamiltonian density:

$$\mathcal{H}_{\mathrm{el}} = \frac{\chi}{4}\left(\frac{d\rho}{dx}\right) + \frac{\chi'}{\rho_0^2}(\rho - \rho_0)^2, \quad (13)$$

where χ and χ' are elastic constants of the cylinder for deformation along the axial and radial directions, respectively. The modified self-dual equations for the magnetic part now read

$$\sin\theta(x)\partial_\Phi\phi = \pm\frac{\rho(x)}{\sqrt{1 + (\partial_x\rho)^2}}\partial_x\theta(x)\,,$$

$$\frac{\sin\theta(x)}{\sqrt{1 + (\partial_x\rho)^2}}(\partial_x\phi) = \mp\frac{\partial_\Phi\theta(x)}{\rho}\,.$$

When considering cylindrically symmetric solutions, these equations reduce to

$$\frac{\sin\theta(x)}{\partial_x\theta} = \pm\frac{\rho}{\sqrt{1 + (\partial_x\rho)^2}}\,. \quad (14)$$

Physically this implies that given a spin distribution $\theta(x)$ one can uniquely determine $\rho(x)$ which would minimize the magnetic part of the Hamiltonian and vice versa. We do not exactly solve the highly nonlinear equation (14), but rather show that with an appropriate choice for $\rho = \rho(x)$ we can minimize the global Hamiltonian $H_{\mathrm{magn+el}}$. We make the following cylindrically symmetric, periodic ansatz for ρ and θ:

$$\rho = \rho_0 - \epsilon\rho_0 \,\mathrm{cn}^2\left(\frac{x}{k\rho_0}, k\right) \quad \text{and} \quad (\theta, \Phi) = \left[\arcsin \mathrm{sn}\left(\frac{x}{k\rho_0}, k\right) + \frac{\pi}{2}, \varphi\right],$$

where cn is the Jacobi elliptic function, cosine amplitude. The periodic magnetic soliton solution and the accompanying periodic pinch of the elastic cylinder are schematically depicted in Fig. 1. Assuming that the quantities $J/\chi\rho_0^2$ and $J/\chi'\rho_0^2$ are small compared to unity, we expand the Hamiltonian density up to third order in ϵ:

$$\rho\sqrt{1+\dot{\rho}^2}\;\mathcal{H}_{\mathrm{el}} = J\left[2\rho_0 \,\mathrm{cn}^2\left(\frac{x}{k\rho_0}, k\right) + \frac{k'^2}{k^2\rho_0}\right] + \epsilon\left[-\frac{Jk'^2}{k^2\rho_0} + \epsilon\rho_0\left(\frac{\chi}{k^2} + \chi'\right)\right]\mathrm{cn}^2\left(\frac{x}{k\rho_0}, k\right) - \epsilon^2 H_\alpha - \epsilon^3 H_\beta - O(\epsilon^4),$$

where H_α and H_β are positive functions.

If we choose $\epsilon = Jk'^2/2[\chi + \chi' k^2]\rho_0^2$, then the total energy for the elastic case is smaller than that for the rigid one. The ansatz for θ does not correspond to the exact spin distribution on the periodically deformed cylinder which reaches the minimum energy for that given geometry because we have chosen $\rho = \rho_0$ in the ansatz for θ. This particular ansatz corresponds to a higher energy of the magnetic soliton. Nevertheless, even with this ansatz, we can lower the total energy, that is, $H_{\mathrm{elastic}} < H_{\mathrm{rigid}}$. In the case $k \to 1$ ($k' \to 0$), $\epsilon \to 0$. Here for a single twist soliton no elastic perturbation of the support lowers the magnetic energy [6]. [The one soliton solution, Eq. (4), satisfies the self-duality equation for the rigid cylinder, Eq. (7).]

In conclusion, we have considered the classical Heisenberg model on an infinite cylinder and found that a periodic topological spin soliton induces a periodic pinch of the cylinder. This is a consequence of the violation of the self-duality equations in contrast with the single soliton case [6]. A pinch can be induced for the single soliton case by introducing spin anisotropy [6]. However, for the periodic case violation of self-duality results from interaction between the solitons. An alternative perspective is that satisfying self-duality amounts to eliminating interaction energy at the cost of elastic deformation. This effect should be observable, with use of ultrasonic techniques, in cylindrically wrapped thin films of magnetic materials, specifically layered 2D Heisenberg magnets such as $(C_nH_{2n+1}NH_3)_2MX_4$ and $[NH_3(CH_2)NH_3]MX_4$ for $n \leq 16$, where M = Cr, Mn, Fe, Cu, Cd and X = Cl, Br [10]. Other examples include K_2CuF_4, Ca_2MnO_4, Rb_2FeF_4, etc. [10], and magnetic Langmuir-Blodgett films of manganese stearate $Mn(C_{18}H_{35}O_2)_2$ [11]. Interestingly, a stable, finite amplitude peristaltic state (periodic shrinking) of tubular fluid membranes was recently observed [12], although for a different physical reason.

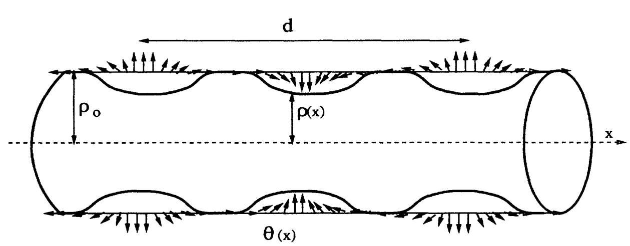

FIG. 1. Cylindrically symmetric magnetic soliton lattice solution and periodic pinch on an elastic cylinder.

R. D. and S. V.-G. acknowledge the hospitality of the Theoretical Division and Center for Nonlinear Studies at Los Alamos National Laboratory. This work was supported by the U.S. DOE.

[1] A. A. Belavin and A. M. Polyakov, JETP Lett. **22**, 245 (1975); S. Trimper, Phys. Lett. **70A**, 114 (1979).
[2] S. Chakravarty, B. Halperin, and D. Nelson, Phys. Rev. Lett. **60**, 1057 (1988).
[3] F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983); Phys. Lett. **93A**, 464 (1983).
[4] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, New York, 1991).
[5] E. B. Bogomol'nyi, Sov. J. Nucl. Phys. **24**, 449 (1976).
[6] S. Villain-Guillot, R. Dandoloff, and A. Saxena, Phys. Lett. A **188**, 343 (1994).
[7] B. Horovitz, Phys. Rev. Lett. **46**, 742 (1981); Phys. Rev. B **35**, 734 (1987).
[8] A. Saxena and A. R. Bishop, Phys. Rev. A **44**, R2251 (1991).
[9] J. Rubinstein, J. Math. Phys. (N.Y.) **11**, 258 (1970).
[10] L. J. de Jongh, *Magnetic Properties of Layered Transition Metal Compounds* (Kluwer Academic Publisher, Dordrecht, 1990).
[11] M. Pomerantz, Surf. Sci. **142**, 556 (1984).
[12] R. Bar-Ziv and E. Moses, Phys. Rev. Lett. **73**, 1392 (1994).