Complex Classical Trajectories and Chaotic Tunneling

Akira Shudo
Institute for Molecular Science, Myodaiji, Okazaki 444, Japan

Kensuke S. Ikeda*
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan
(Received 7 April 1994)

Tunneling phenomena in the presence of chaos are investigated. Several remarkable features, which form a sharp contrast to the tunneling in the integrable system, can be well understood in terms of the semiclassical theory including complex classical trajectories. In particular, it is found that the dominant contribution to the tunneling regime comes from many complex branches which are not connected with any real manifolds and are linked at caustics to form the bifurcation chains. Chaotic tunneling is proposed as a new class of tunneling phenomena which originate from the complicated nature of the complex dynamical system.

PACS numbers: 05.45.+b

An underlying idea of the semiclassical theory is not only to provide us with an approximate solution to the Schrödinger equation but to describe or understand wave phenomena in terms of language of rays or classical trajectories [1]. Indeed one can obtain a clear picture of quantum-classical correspondence for integrable systems to which the semiclassical theory is successfully applied. Although one encounters serious difficulties once the semiclassical theory is applied to chaotic systems [2], optimistic scenarios to overcome them, which rely on our detailed knowledge of *real* classical orbits, are recently proposed [3].

On the other hand, as already pointed out more than 20 years ago, in addition to real classical trajectories *complex* trajectories are also needed to describe quantum mechanics more properly [4]. In fact, the transition to classically inaccessible regions can be realized only by including complex classical trajectories. Tunneling is a typical example of such a purely quantum mechanical phenomenon. An attempt to utilize complex trajectories has been actually undertaken in the field of chemical reactions, and it has been shown that classical S-matrix theory is improved by including complex trajectories [5].

Although the necessity of complex trajectories is recognized and generic systems are known to show chaotic behavior, the nature of tunneling and the role of complex classical orbits in classically nointegrable systems are still unclarified. The purpose of the present Letter is, therefore, to aim at a semiclassical understanding of tunneling phenomena in the presence of chaos in terms of complex classical trajectories. In particular, we propose a new class of tunneling phenomena we call *chaotic tunneling,* which originates from complicated natures of complex classical trajectories.

The model system we use here is designed to extract the tunneling effect as purely as possible and to avoid the confusion caused by the transition due to the real classical trajectories. We consider the following kicked rotor system:

$$H = H_0(\hat{p}) + V(\hat{\theta}) \sum_n \delta(t - n), \tag{1}$$

where

$$H_0(p) = \frac{p^2}{2} \frac{(p/p_D)^6}{(p/p_D)^6 + 1} + \omega p, \tag{2}$$

$$V(\theta) = K \sin\theta. \tag{3}$$

In the bounded region $|p| < p_D$, this system is approximately equivalent to the kicked linear oscillator $[H_0(p) = \omega p]$, which means that there exists a Kolmogorov-Arnold-Moser (KAM) band whose width is controlled by p_D. Outside the region $|p| \gg p_D$, it tends to the well-known standard mapping $[H_0(p) = p^2/2]$. A phase space portrait for $K = 1.2$ and $p_D = 5$ is depicted in Fig. 1(a).

Throughout the present analysis we take the time-domain approach [6] and first observe the time evolution of a quantum state which is the momentum eigenstate with $p_0 = 0$ included in the KAM band. In the very early stage of the time evolution ($t = 1-3$), the quantum mechanical

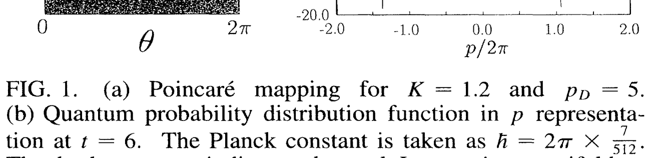

FIG. 1. (a) Poincaré mapping for $K = 1.2$ and $p_D = 5$. (b) Quantum probability distribution function in p representation at $t = 6$. The Planck constant is taken as $\hbar = 2\pi \times \frac{7}{512}$. The broken curve indicates the real Lagrangian manifold at $t = 6$. The arrows inserted show the positions of "cliffs."

0031-9007/95/74(5)/682(4)$06.00

tunneling occurs only from the positions where the real Lagrangian manifold is folded. The real Lagrangian manifold we use here is a set of classical trajectories which initially satisfy the conditions $p_0 = 0$ and $\theta_0 \in [0, 2\pi]$. These points on the Lagrangian manifold just correspond to caustics, and tunneling in the very initial stage can be explained as the penetration starting from caustics. This mechanism is essentially the same as tunneling in the integrable system. As is shown below, a simple interpretation using complex classical trajectories is possible.

However, as time proceeds, more complicated tunneling patterns become visible. Indeed, as displayed in Fig. 1(b) the appearance of the amplitude distribution $|\Psi(p,t)|^2$ at $t = 6$ does not seem to allow us the simple interpretation mentioned above. We immediately notice the following remarkable features in the tunneling regime: (1) As p goes out of the range occupied by the real Lagrangian manifold [see Fig. 1(a)], the probability amplitude decays very rapidly through the KAM regular regime at the rate expected for the completely integrable linear oscillator, i.e., $p_d = \infty$, indicated by the dashed curves in Fig. 1(b). (2) The initial rapid decay soon changes to slower decay beyond values of p corresponding to the boundary between the integrable region and the chaotic sea. (3) As p goes further into the chaotic sea, the probability amplitude forms plateaus, which are accompanied by complicated oscillations. (4) The amplitude again decays with several "cliffs" (indicated by arrows in the figure) located at apparently random positions. Extensive numerical study reveals that such features are common at arbitrary time evolution steps. Thus, one can regard the features listed are typical at least for the present model.

The semiclassical wave function in the momentum representation is expressed as a sum over classical trajectories which satisfy the boundary conditions, $p(0) = p_0$ and $p(t) = p_t$,

$$\Psi(p_t, t) = \sum_k A_k \exp[iS_k(p_t, p_0, t)/\hbar + \phi_k], \tag{4}$$

where each trajectory is labelled by the index k. A_k denotes the amplitude factor. S_k is the classical action along the classical trajectory obtained by solving Eqs. (2) and (3) in the extended phase space ($-\infty \le \theta \le \infty$) [7],

$$S_k = \sum_{j=1}^{t} \{H_0(p_{j,k}) + V(\theta_{j,k})\} - \sum_{j=1}^{t} (p_{j,k} - p_{j-1,k})\theta_{j,k}, \tag{5}$$

and ϕ_k represents the phase correction associated with the Maslov index.

One efficient method to describe the tunneling process is to take into account complex classical trajectories which are introduced by analytically continuing the dynamical variables into the complex plane. The analytical continuation in the present case is carried out by extending the initial angle θ_0 to the complex domain, i.e., $\theta_0 = \xi + i\eta (\xi, \eta \in \mathbf{R})$, with the initial and final momentum real valued. By solving such a kind of shooting problem, we have actually found an enormous number of candidates which can contribute to the semiclassical wave function. Figure 2(a) gives a set M_t of initial Lagrangian manifolds whose initial and final momenta take both real values, i.e.,

$$M_t = \{(\xi, \eta) | p_0 = 0,\ \mathrm{Re}\, p_t(\xi + i\eta) \in \mathbf{R},\ \mathrm{Im}\, p_t(\xi + i\eta) = 0\}. \tag{6}$$

Since we now fix the initial momentum as $p_0 = 0$, the final momentum p_t is a unique analytical function of θ_0. Therefore, if we choose an arbitrary initial point on the complex Lagrangian manifolds M_t and iterate it by t steps, then we necessarily reach a certain point with a real final momentum, but its explicit value p_t cannot be read from Fig. 2(a).

In Fig. 2(a), the horizontal axis $\eta = 0$ represents the real Lagrangian manifold which contributes to the semiclassical wave function but does not escape from KAM band regime. Thus, the tunneling observed in Fig. 1(b) is only described by complex Lagrangian manifold. The two branches connecting with the real Lagrangian manifold ($\eta = 0$) at the caustics are complex branches which describe the feature (1), i.e., the first rapid decay through KAM integrable regime. We call these complex branches

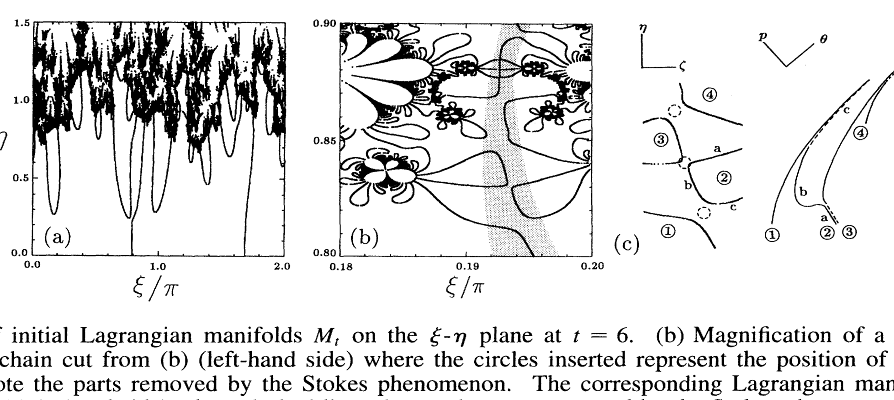

FIG. 2. (a) A set of initial Lagrangian manifolds M_t on the ξ-η plane at $t = 6$. (b) Magnification of a small region on ξ-η plane. (c) A Laputa chain cut from (b) (left-hand side) where the circles inserted represent the position of the complex caustics and dashed lines denote the parts removed by the Stokes phenomenon. The corresponding Lagrangian manifolds projected onto the $(\mathrm{Re}\,\theta_{t+1}, p_t)$ plane (right-hand side) where dashed lines denote the parts removed by the Stokes phenomenon.

natural complex branches: They give dominant contribution to tunneling if the system is completely integrable, i.e., $p_D \to \infty$. In the integrable limit, the natural branches yield the rapid decay indicated by the dashed lines in Fig. 1(b). These branches are also the origin of tunneling observed in the very initial stage.

Natural branches reproduce the penetration through the integrable region, but they are not sufficient to explain the remaining features (2)–(4). The origin should be, therefore, attributed to the infinitely many self-similar branches which are no more connected with the real Lagrangian manifold. We call such complex manifolds which float in the large imaginary domain of complex θ_0 plane *Laputa branches* which are named after the floating island appearing in the famous story *Gulliver's Travels.*

Among these enormous numbers of Laputa branches, we have adopted a heuristic or semiempirical criterion that only the Laputa branches having relatively small imaginary parts of θ_0 are taken into account. This is based on the observation that if the imaginary part of the initial angle is small, then the imaginary part of the classical action which governs the amplitude factor of each contribution is also small. Indeed, slow decay of the wave function described in (2) is attributed to large scale branches hanging down from a cloud of Laputa branches [see Fig. 1(a)] [8]. Various features of the wave function described in (3) and (4), however, cannot be explained without considering detailed structures of the Laputa branches in the cloud.

We show in Fig. 2(b) the magnification of the region where a large scale branch disappears into the cloud. In most cases, a Laputa branch folds to form a petal of a self-similar flowerlike object. In some cases, a Laputa branch extends between different flowerlike objects. Surprisingly, the final momentum p_t ranges from $-\infty$ to $+\infty$ as ξ-η moves along each branch. This fact seems to suggest that each branch contributes to the wave function at all values of p. However, not all of them contribute significantly to the wave function: It is found that the sequences of Laputa branches running through the channel between flowerlike structures indicated by the hatched region crucially contribute to the wave function, giving rise to features (3) and (4). We call such a sequence *Laputa chain.*

We describe the relationship among the contributions to the wave function from different constituent branches of a Laputa chain of Fig. 2(c) cut out from Fig. 2(b). Figure 3(a) depicts the corresponding semiclassical wave functions for the three successive branches taken from the Laputa chain. A striking feature is that the wave function on a branch, say 2, decreases in the opposite direction to the first wave function. Similarly, decaying character of the wave function given by the branch 4 is opposite to that of the branch 3. Namely, wave functions with reversed decaying character appear alternately along a single Laputa chain.

We present here more detailed descriptions based on our numerical observations. Note that each of the constituent branches is composed of three characteristic sections labeled by a, b, and c. Each wave function decays rapidly and then forms a plateau. It suddenly decays again, forming a cliff. On the ξ-η plane, the steep gradient of the wave function corresponds to the section c, where the imaginary part of the action $\mathrm{Im}S_k$ takes a negative large value. Such an explosion of probability amplitude is unphysical and should be removed. This operation is justified from the following observations. The branch 2 approaches close to the neighboring branch 1 at the joint between two sections b and c. Between the two branches there is a complex caustic whose final momentum p_t^* has a nonzero imaginary part, and its projection onto the ξ-η plane is shown in Fig. 2(c) as a circle. It is at this p_t^* that the contributions to the wave function from the branches 1 and 2 cross with each other, exhibiting reversed decaying behaviors as is shown in Fig. 3(a). Close to the caustics, there will be a branch exhibiting exponential explosion which should be unphysical and cutoff because of the Stokes phenomenon [9]. The branch section c is just such an unphysical one. To locate the position where the unphysical branch section is cut off, we have to know the precise locations of the Stokes line. Unfortunately, they cannot be determined because of the lack of a theory of higher-dimensional Stokes phenomenon [10]. As a working hypothesis, we cut off the unphysical branch section when the sign of $\mathrm{Im}S_k$ changes from positive to negative. This hypothesis works very well to reproduce the fully quantum mechanical wave function as will be demonstrated later. Much improvement is gained if one employs the principle of exponential dominance [11].

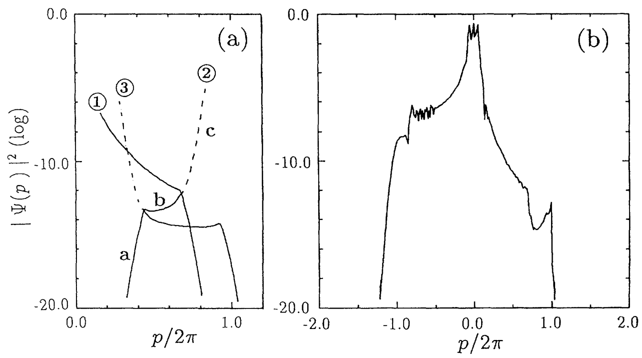

FIG. 3. (a) Individual semiclassical probability distributions, each of which corresponds to the Laputa branch shown in Fig. 2(c). The dashed lines represent the parts cut off by the Stokes phenomenon. (b) Semiclassical probability distribution obtained by summing up all dominant contributions. All parameters are the same as in Fig. 1(b).

The Laputa branch 2 again collides with the next branch 3, where $\mathrm{Im}S_k$ begins to decay rapidly and the branch contributing significantly to the semiclassical wave function is interchanged. In other words, the carrier of tunneling probability contribution switches to the new branch 3. As demonstrated in Fig. 3(a), the direction of increase of the wave function as a function of p_t is opposite, which means that if we change to the new branch, the corresponding p_t moves in the opposite direction.

A similar collision occurs between the new branch 3 and the next generation 4, and its contribution to the wave function varies with p_t in an opposite manner. In this way, the switching of the contributing wave function continues along a Laputa chain. On the right-hand side of Fig. 2(c) we depict the Lagrangian manifold projected onto the $(\mathrm{Re}\,\theta_{t+1}, p_t)$ plane which corresponds to the Laputa chain shown on the left-hand side of Fig. 2(c). On this plane, the caustics are located where two branches come close together. If one traces the sections of Laputa chain shown in the left-hand side of Fig. 2(c) contributing to the plateaus, one can find that its counterpart on the $(\mathrm{Re}\,\theta_{t+1}, p_t)$ plane forms a stretched object folded at caustics. This is the manifestation of the chaos in the tunneling regime.

Erratic oscillatory structures observed in the purely quantum wave function is explained by the violent interference among many Laputa branches whose plateau regions have similar probability amplitude. As the Laputa chain goes more deeply into the deep imaginary domain on the ξ-η plane, the imaginary part of caustics between adjacent branches gradually increases, which results in the contribution from the Laputa branches with larger η of higher generation becoming smaller. This is explained by the fact that the slope of wave function around the complex caustics is determined by its imaginary part, i.e., $\mathrm{Im}\,p_t^*$ or $\mathrm{Im}\,\theta_{t+1}^*$ according to the relation $\partial \mathrm{Im}\{S(p_0, p_t)\}/\partial p_t = -\mathrm{Im}\,\theta_{t+1}$. The petal of the flowerlike structures other than those composing the Laputa chain is accompanied by complex caustics with extremely large imaginary part and thus does not contribute to wave function because of the same reason mentioned above.

In addition to this gradual decrease of the plateau height, one more remarkable fact is that the location of the cliff moves quite irregularly as the generation changes. Several cliffs observed in Fig. 1(b) are, therefore, a consequence of such erratic fluctuation of the switching points.

The above scenario generating the complicated tunneling is built only on the presence of a single Laputa chain. Further complication is added when other Laputa chains are taken into account. In fact, besides the Laputa chain illustrated in Fig. 2(b), several other chains have also been discovered in search of all possible Laputa branches. Contributions from such Laputa chains may dominate the other Laputa chain in a certain region. This gives rise to more complicated aspects of tunneling phenomenon.

As time proceeds, the number of Laputa branches grows at a considerably rapid rate, and the lower bound of the cloud of Laputa branches on the ξ-η plane gradually comes down. Along the Laputa chains persistent at arbitrary steps the number of successive branches also increases, and the caustics between Laputa branches approach the real plane. As a result, the height difference between successive plateaus becomes smaller, which leads to much more erratic interference on them and the creation of new cliffs.

In any case, the successive bifurcation is the origin of *chaotic tunneling*. Indeed, as shown in Fig. 3(b), constructing the semiclassical wave function by including all possible Laputa branches, we can well produce various aspects in the tunneling region. The disagreement detected in the oscillating structures on the plateau would be mainly due to inappropriate location of the Stokes lines.

In summary, we have demonstrated that it is impossible to understand the tunneling in the presence of chaos without complex classical trajectories. In particular, the complex branches having no connection with the real manifold, what we have called Laputa branches, play a crucial role to generate chaotic tunneling. The present result strongly suggests the importance of the complex trajectories which have not been taken into consideration in the analysis of classically chaotic systems in terms of the semiclassical theory [11]. In order to describe the quantum system with the mixed phase space, deeper understandings of the complex dynamical system are strongly desired.

The authors appreciate very much P. Gaspard for stimulating discussions. This work is supported by a Grant-in-Aid for Scientific Research on Priority Areas, "Computational Physics as a New Frontier in Condensed Matter Research," from the Ministry of Education, Science and Culture, Japan.

*Present address: Faculty of Science and Engineering, Ritsumeikan University, Noji-cho 1916, Kusatsu 525, Japan.

[1] M. V. Berry and K. E. Mount, Rep. Prog. Phys. **35**, 315 (1972).
[2] A. Voros, J. Phys. A **21**, 685 (1988); B. Eckhardt, and E. Aurell, Europhys. Lett. **9**, 509 (1989).
[3] P. Cvitanovic and B. Eckhardt, Phys. Rev. Lett. **63**, 823 (1989); M. V. Berry and J. Keating, J. Phys. A **23**, 4839 (1990); M. Sieber and F. Steiner, Physica (Amsterdam) **44D**, 248 (1990).
[4] R. Balian and C. Bloch, Ann. Phys. (N.Y.) **85**, 514 (1974); A. Voros, Ann. Inst. Henri Poincaré **39**, 211 (1983).
[5] W. H. Miller and T. F. George, J. Chem. Phys. **56**, 5668 (1972).
[6] S. Tomsovic and E. J. Heller, Phys. Rev. Lett. **67**, 664 (1991).
[7] K. Ikeda, Ann. Phys. (N.Y.) **227**, 1 (1993).
[8] A. Shudo and K. Ikeda, Prog. Theor. Phys. Suppl. (to be published).
[9] See, for example, J. Heading, *An Introduction to Phase-Integral Methods* (Metheun, London, 1962).
[10] A working hypothesis which is a natural extension of the principle of exponential dominance has been proposed in S. Adachi, Ann. Phys. (N.Y.) **195**, 45 (1989).
[11] A. Shudo and K. S. Ikeda (to be published).