Gaussian Orthogonal Ensemble Statistics in a Microwave Stadium Billiard with Chaotic Dynamics: Porter-Thomas Distribution and Algebraic Decay of Time Correlations

H. Alt,[1] H.-D. Gräf,[1] H. L. Harney,[2] R. Hofferbert,[1] H. Lengeler,[3] A. Richter,[1] P. Schardt,[1] and H. A. Weidenmüller[2]

[1]*Institut für Kernphysik, TH Darmstadt, D-64289 Darmstadt, Germany*

[2]*Max-Planck-Institut für Kernphysik, D-69029 Heidelberg, Germany*

[3]*AT-Division, CERN, CH-1211 Geneva 23, Switzerland*

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The complete set of resonance parameters for 950 resonances of a superconducting microwave cavity connected to three antennas has been measured. This cavity simulates the quantum mechanics of a particle in a Bunimovich stadium. The partial widths are found to follow a Porter-Thomas distribution. The Fourier transforms of the S-matrix autocorrelation functions decay algebraically (nonexponentially) in time. These results agree perfectly with the predictions of random-matrix theory. They constitute one of the most stringent tests ever of this expected connection between chaotic dynamics and random-matrix theory.

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Quantum manifestations of classical chaos have received much attention in recent years. The spectral fluctuation properties of systems, which are fully chaotic in the classical limit, were investigated both analytically and numerically [1]. Generically, it was found that these properties coincide with those of the random matrix ensemble having the proper symmetry. (The caveats which have to be attached to this statement are irrelevant for what follows and are not discussed here.) For time-reversal invariant systems, the relevant ensemble is the Gaussian orthogonal ensemble (GOE).

Most of the information available today on spectral fluctuation properties relates to the statistics of *energy levels,* while much less information is available on the statistics of *wave functions* or *decay amplitudes*. Perhaps statistically the most convincing evidence so far of the latter type has been presented in Ref. [2], where the partial width amplitudes of about 400 proton resonances feeding three different exit channels were analyzed and shown to agree with GOE predictions.

In this Letter, we present and analyze experimental results on the decay amplitudes of 950 resonances in a superconducting microwave stadium billiard coupled to three antennas, yielding three partial width amplitudes per resonance. Both the quality of our data and the statistical accuracy afforded by the large number of resonances make this data set, in our opinion, the most comprehensive test yet for GOE fluctuation properties of decay amplitudes and, hence, wave functions in a classically chaotic system. One of the GOE tests, suggested in Ref. [3], shows for the first time that on average, the decay in time of the resonances is not exponential but algebraic.

The stadium billiard is known [4] to be fully chaotic in the classical limit. Moreover, the quantum case can be simulated [5] electromagnetically in terms of a sufficiently flat microwave resonator. We use the superconducting niobium cavity as described in Ref. [6] (see inset in Fig. 1). In our earlier experiment [6], attention was focused on the resonance *frequencies,* and the coupling to the antennas therefore minimized. Now, we are interested in *decay widths*. In comparison with Ref. [6], we have therefore increased the coupling between the cavity and the antennas by about a factor of 10. Then, the widths of the resonances (which are proportional to the square of the eigenfunction at the respective location of the antennas) are dominated by the microwave intensity fed into the antennas: Dissipation by the walls of the cavity contributes only 5% to the average total width of the resonances. This low value can be attained only in a superconducting cavity and is crucial for the statistical accuracy of our results. Thus we are essentially dealing with a scattering problem described by a 3×3 matrix.

The upper part of Fig. 1 shows the absolute value of S_{11}, i.e., of the $(1,1)$ element of the scattering matrix versus frequency. We display only a small part of the total frequency interval used in the experiment. Very narrow resonances are seen to interfere destructively with a slowly varying wavy background. The latter is due to intensity attenuation in the cables connecting the antennas with the network analyzer. Indeed, without such attenuation we would have $|S_{ab}| = \delta_{ab}$ outside the resonances. We model the influence of the three cables on the transmitted wave in terms of three complex phase shifts δ_a, with $a = 1$, 2, and 3. The data show very clearly that in the vicinity of a resonance, we are permitted to use the single-level Breit Wigner formula. For frequencies ω close to a resonance labeled μ with frequency ω_μ, we therefore write the scattering matrix in the effective form

$$S_{ab}^{\mathrm{eff}}(\omega) = e^{i\delta_a}\left(\delta_{ab} - i\frac{\Gamma_{\mu a}^{1/2}\ \Gamma_{\mu b}^{1/2}}{\omega - \omega_\mu + \frac{i}{2}\Gamma_\mu}\right)e^{i\delta_b}, \qquad (1)$$

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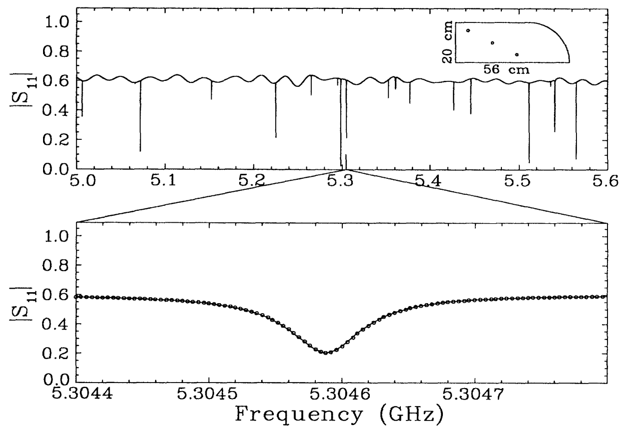

FIG. 1. The S-matrix element S_{11} in the frequency range of 5.0 to 5.6 GHz (upper part) not corrected for absorption in the cables (background). The density of data points can be seen from the blown-up resonance structure in the lower part of the figure. The solid line shows a Lorentzian fit which determines the resonance parameters.

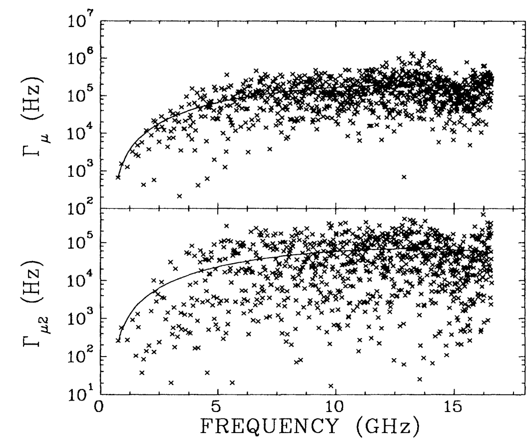

FIG. 2. Total widths (upper part) and partial widths of channel 2 (lower part) are plotted versus resonance frequency. The secular frequency dependence is given by the solid line (see text).

where

$$\Gamma_\mu = \sum_{c=1}^{3} \Gamma_{\mu c} + \Gamma_{\mu,\mathrm{wall}} . \tag{2}$$

The dissipation in the walls of the superconducting cavity is represented by $\Gamma_{\mu,\mathrm{wall}}$. In writing Eq. (1) with real decay amplitudes $\Gamma_\mu^{1/2}$, we have assumed that the microwave intensity is only *attenuated* and not *reflected* in the cables. For the first 950 resonances, i.e., up to a frequency of 16.85 GHz, we have avoided reflection by a careful choice of the cables and connectors. At each resonance, data have been taken with a step size that varied between 100 Hz/step and 4 kHz/step depending on the total width of the resonance, cf. the lower (magnified) part of Fig. 1. The solid line is a fit to the data points using Eq. (1). All resonance curves agree to very high precision with Lorentzians [7]. The resonance parameters $\Gamma_{\mu c}$, Γ_μ, and ω_μ and the background parameters δ_c were determined from magnitudes and phases of the three elastic scattering matrix elements $S_{aa}^{\mathrm{eff}}, a = 1$, 2, and 3. The inelastic S-matrix elements were found to be consistent with this analysis. From the three values of Γ_μ obtained independently for each μ, we estimate the error in the resonance parameters to be about 5%.

Figure 2 shows a plot of the total widths Γ_μ (upper part) and of the partial widths $\Gamma_{\mu 2}$ (lower part) versus resonance frequency. The data are seen to fluctuate randomly about a slow secular variation with frequency. The latter has been determined by fitting a polynomial (solid lines) of 5th order to the total widths and is caused by the fact that the coupling of the antennas to the cavity varies slowly with frequency. For the statistical analysis of the fluctuations, the secular variation has been removed by scaling all widths with this polynomial. With the average total width of 154 kHz normalized to unity, and with brackets denoting the average over all 950 resonances, we find $\langle \Gamma_{\mu 1} \rangle = 0.20$, $\langle \Gamma_{\mu 2} \rangle = 0.35$, and $\langle \Gamma_{\mu 3} \rangle = 0.40$. This yields $\langle \Gamma_{\mu,\mathrm{wall}} \rangle = 0.05$.

The GOE predicts a Gaussian distribution for the decay amplitudes $\Gamma_{\mu c}^{1/2}$ or, equivalently, a χ^2 distribution with the number of degrees of freedom (ν) equal to 1 (this is commonly referred to as a Porter-Thomas [8] distribution) for the $\Gamma_{\mu c}$ with fixed c. We have fitted the measured distributions for the $\Gamma_{\mu c}$ optimizing ν and find $\nu = 1.06 \pm 0.09$, 1.07 ± 0.09, and 1.05 ± 0.09 for $c = 1$, 2, and 3, respectively. Figure 3 shows the distribution of the measured and scaled $\Gamma_{\mu 2}$. We have also tested the GOE prediction that the decay amplitudes for different channels are uncorrelated. (This obviously requires that the distances between neighboring antennas are larger than the maximum wavelength used in the experiment. For this reason we had to leave out the first 12 resonances in the analysis of the correlation functions.) The normalized cross-correlation coefficients are found to be $C(\Gamma_1, \Gamma_2) = 0.01 \pm 0.05$, $C(\Gamma_1, \Gamma_3) = 0.07 \pm 0.07$, and $C(\Gamma_2, \Gamma_3) = 0.07 \pm 0.06$, consistent with uncorrelated amplitudes. If we neglect $\Gamma_{\mu,\mathrm{wall}}$, the distribution of Γ_μ should have $\nu = 3$ in the case of all three average partial widths $\langle \Gamma_{\mu c} \rangle$ being equal. Since the $\langle \Gamma_{\mu c} \rangle$ are unequal, this expectation for ν is reduced to $\nu = 2.79$. Our best fit is $\nu = 2.82 \pm 0.24$. The GOE predicts that the resonance widths are statistically independent from the resonance energies. To apply this test, it is necessary to unfold the spectrum [6]. The experimental result $C(\Gamma, \omega) = 0.02 \pm 0.05$ agrees with this prediction.

As a further test of GOE properties, we have calculated the autocorrelation function of the elastic S-matrix elements versus frequency. To this end, we use the resonance parameters determined above and construct the S

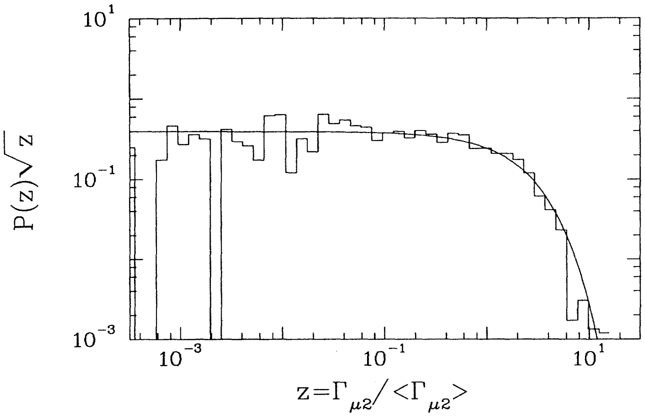

FIG. 3. Distribution of $\Gamma_{\mu 2}/\langle\Gamma_{\mu 2}\rangle$ on a doubly logarithmic scale. The solid line corresponds to a Porter-Thomas distribution.

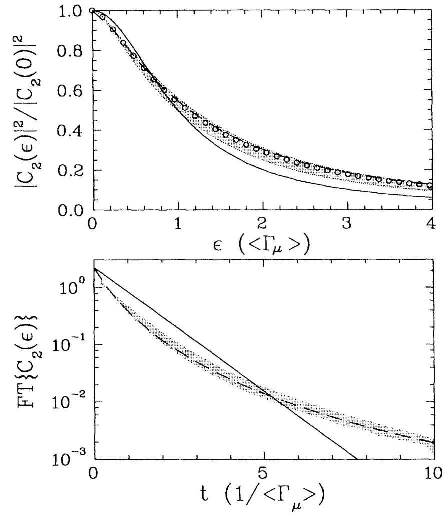

FIG. 4. Upper part: Autocorrelation function (circles) $C_2(\epsilon)$; the shaded band indicates the errors. The dashed and the solid lines represent the GOE prediction and a Lorentzian, respectively. Lower part: Fourier transform of $C_2(\epsilon)$, errors indicated by the shaded band. The full line is the Fourier transform of the Lorentzian, and the dashed curve is the GOE prediction.

matrix *of the cavity* (i.e., without the cables) from the expression

$$S_{ab}(\omega) = \delta_{ab} - i \sum_{\mu} \frac{\Gamma_{\mu a}^{1/2} \quad \Gamma_{\mu b}^{1/2}}{\omega - \omega_\mu + \frac{i}{2}\Gamma_\mu}. \qquad (3)$$

From this expression, we have calculated the autocorrelation functions

$$C_c(\epsilon) = \overline{S_{cc}(\omega)S_{cc}^*(\omega + \epsilon)} - |\overline{S_{cc}}|^2 \qquad (4)$$

for $c = 1, 2,$ and 3. Here, the overbar denotes the average over ω. The result for $c = 1$ is shown as circles in the upper part of Fig. 4. Since the individual experimental points are highly correlated, we could not attribute error bars to these points individually in a meaningful way. We have used instead the method described by Efron [9]. We have considered the experimental set of 5×938 resonance parameters ($\Gamma_{\mu 1}$, $\Gamma_{\mu 2}$, $\Gamma_{\mu 3}$, Γ_μ, ω_μ) as the ensemble from which to construct random scattering matrices. Hence, 938 parameter 5-vectors were drawn at random from the ensemble (with the possibility of having the same vector drawn several times). This set was used in Eqs. (3) and (4). Repeating this procedure 10 times we found that the functions $C_c(\epsilon)$ are restricted to the shaded band.

The shape of the shaded band differs markedly but not unexpectedly [10] from that of a Lorentzian with width $\langle\Gamma_\mu\rangle$ shown as a solid line in the upper part of Fig. 4. The contribution of each individual resonance to the correlation functions (4) is, of course, Lorentzian in shape. However, different resonances contribute Lorentzians of *different* widths, so that the average over all resonances does not have Lorentzian shape. This effect can be seen only if the total width shows marked fluctuations. Hence, (i) the number M of open channels must be small, and (ii) the ratio $\langle\Gamma_{\mu,\mathrm{wall}}\rangle/\langle\Gamma_\mu\rangle$ must be small compared to unity. In the microwave experiment of Ref. [11]—performed at room temperature—condition (ii) was violated. The data did not display the non-Lorentzian line shape [12] although condition (i) was well satisfied ($M =$ 1). The non-Lorentzian shape is a quantum phenomenon. Indeed, in the semiclassical approximation, we have $M \gg 1$ and purely exponential decay. Moreover, in the classical limit a chaotic billiard with a small hole shows exponential time decay, as expected from ergodicity. For the Sinai billiard, this was shown in Ref. [13]. For the stadium, we have verified it numerically.

The result displayed in the upper part of Fig. 4 is quantitatively compatible with the GOE model [14] for resonance reactions. In the limit of isolated resonances this model yields [cf. Eq. (8) of Ref. [15]]

$$C_c(\epsilon) = \frac{3}{2}\int_0^\infty dx \left(\frac{T_c}{1 + T_c x}\right)^2 \exp\left(-i\pi\frac{\epsilon}{d}x\right)\Pi. \qquad (5)$$

Here, d is the mean level distance. For isolated resonances, the "transmission coefficients" are given by $T_c = 2\pi\langle\Gamma_{\mu c}\rangle/d$. The symbol Π stands for $\prod_{e=1}^{3}(1 + T_e x)^{-1/2}$.

In the analysis of the level correlation function for the stadium billiard performed in Ref. [6], the "bouncing ball orbits" of the stadium played a significant role: These orbits modulate the average level density on a scale of several mean level spacings d. In the S-matrix correlation functions displayed in Fig. 4, the bouncing ball orbits are conspicuously absent. This is because these correlation functions are significantly different from zero only in an interval much smaller than d.

It is instructive to investigate the Fourier transform $\mathcal{F}_c(t)$ of the autocorrelation function $C_c(\epsilon)$ defined in Eq. (5),

$$\mathcal{F}_c(t) = 3\frac{2\pi}{d}\langle\Gamma_{\mu c}\rangle^2(1 + 2\langle\Gamma_{\mu c}\rangle t)^{-2}\Pi', \qquad (6)$$

with Π' given by $\prod_{e=1}^{3}(1 + 2\langle\Gamma_{\mu e}\rangle t)^{-1/2}$. The factor 3 on the right-hand side of Eq. (6) is the elastic enhancement factor [16]. It is specific for the GOE. Inclusion of the power dissipation in the cavity walls tends to improve the agreement. This is not shown because the number of (ficticious) channels needed to account for this process cannot be determined unambiguously [12]. We note that the factor 3 is confirmed by the data. The results for $c = 1, 3$ are very similar to the ones for $c = 2$.

The function $\mathcal{F}$ has a simple interpretation. Suppose that we were to excite all 938 resonances in the cavity *simultaneously* with a very short microwave pulse. Then, $\mathcal{F}_c(t)$ is the derivative of the function which describes the decay in time of that system [3,17]. The function (6) decays *algebraically* and not exponentially, with a characteristic decay time $\langle\Gamma_\mu\rangle^{-1}$ defined in terms of the average strength of the coupling of the resonances to all three antennas. In this sense, it is fair to say that we have for the first time "observed" the nonexponential decay of a quantum system with chaotic classical dynamics. This nonexponential decay is caused by the fluctuations of the total widths. This is seen by an argument similar to the one used above to explain the non-Lorentzian shape in the upper part of Fig. 4.

In summary, we have presented data on the statistics of partial widths for the stadium billiard. All tests applied—Porter-Thomas distribution, lack of correlations among partial widths, lack of correlations between partial widths and resonance energies, non-Lorentzian decay of the S-matrix autocorrelation function, and the related nonexponential time decay of the system—are in perfect agreement with GOE predictions for the statistics of eigenfunctions. Because of the accuracy and the number of our data points, we believe that these results constitute one of the strongest tests ever for the assertion [1] that upon quantization, a classically chaotic system respecting time-reversal symmetry attains GOE fluctuation properties.

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