Replica Symmetry Breaking Instability in the 2D XY Model in a Random Field

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We study the 2D vortex-free XY model in a random field, a model for randomly pinned flux lines in a plane. We construct controlled renormalization group recursion relations which allow for replica symmetry breaking (RSB). The fixed point previously found by Cardy and Ostlund in the glass phase $T < T_c$ is *unstable* to RSB. The susceptibility χ associated to infinitesimal RSB perturbation in the high-temperature phase is found to diverge as $\chi \propto (T - T_c)^{-\gamma}$, when $T \rightarrow T_c^+$. This provides analytical evidence that RSB occurs in finite dimensional models. The physical consequences for the glass phase are discussed.

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One of the most fruitful methods to study disordered systems is the replica method [1] which allows one to average over disorder by introducing n coupled copies of the system. Since it reduces the problem to a translationally invariant system, this method has the considerable advantage, in principle, to allow the use of standard field theory techniques. However, the limit $n \rightarrow 0$ has to be taken, a rather subtle procedure. Indeed, as is well known from the large body of works on spin glasses [2] that the free energy functional is in general minimized, in a glass phase, by a solution which breaks replica symmetry spontaneously, in the hierarchical manner discovered by Parisi [3]. Up to now, however, spontaneous replica symmetry breaking (RSB) has been clearly demonstrated only when a saddle point method could be used naturally. This is the case in either infinite-range or mean-field models where the saddle point method provides the exact solution of the problem and RSB is ubiquitous [4], or, as found more recently [5], in field theories using a Hartree or Gaussian variational method (GVM) which becomes exact in the limit of a large number of components, in some as yet unelucidated sense. As explicitly constructed in both cases the physics of RSB corresponds to systems breaking up in many "states" [4]. Extensions of RSB to finite dimensional systems for which such mean-field theory is not exact have been attempted [6]. However, it is far from obvious, as argued in [7], that it is relevant in real, i.e., low dimensional physical systems.

Another extensively tried method is the renormalization group (RG). There replicas have been used so far simply as a trick to eliminate disconnected graphs in perturbation theory [8]. This procedure, however, implicitly assumes replica symmetry before taking the limit $n \rightarrow 0$. Although the RG treats fluctuations exactly and is therefore more accurate than mean-field theory, there is a risk that it will miss the physics associated to RSB. Thus one would like to link the two methods.

A good model to look for a RG that allows RSB is the two-dimensional XY model in a random field and without vortices (i.e., with spins $S(x) = e^{i\psi(x)}$, where $\psi(x)$ is real valued in $[-\infty, \infty]$). It is particularly interesting because it is one of the simplest nontrivial models of a "glass" in finite dimension, to which several analytical methods can be applied [9–14]. It describes several physical disordered systems, such as randomly pinned flux arrays in a plane [13,15,16], the surface of crystals with quenched bulk or substrate disorder [17], planar Josephson junctions [18], and domain walls in incommensurate solids. In a pioneering work [10], Cardy and Ostlund (CO) studied this model using the RG. They used replicas, introduced n coupled XY models, mapped them onto a Coulomb gas with $n(n-1)/2$ types of vector charges, and constructed the RG equations. They then took the limit $n \rightarrow 0$ thus implicitly assuming replica symmetry. The resulting RG equations, valid near T_c, possess a nontrivial perturbative fixed point for $T < T_c$ at weak disorder $g = g^* \propto T_c - T$. CO concluded that a glass phase exists, controlled by this fixed point. In this phase, one coupling constant flows to infinity, a rather peculiar feature. These results were extended in [17,19], and the disorder averaged correlation function $C(x) = \overline{\langle \psi(x) - \psi(0) \rangle^2}$ was found to grow as $C(x) \sim B(\ln|x|)^2$, faster than $C(x) \sim T \ln|x|$, which holds in the high-temperature phase and for the pure system.

Although there is presently agreement that a glass phase exists in this model for $T < T_c$, its physical properties remain controversial, despite the large number of analytical and numerical studies. Two recent numerical simulations [20,21] have shed some doubt on the Cardy-Ostlund RG. Both were found incompatible with the $C(x) \sim B(\ln|x|)^2$ prediction. In the dynamics [20] it was found that the velocity develops a nonlinear dependence on the driving force for $T < T_c$, giving evidence of a glassy phase, but with an exponent inconsistent with the one of a conventional dynamic RG study [11]. These simulations,

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however, seem compatible with our previous analytical results using the GVM [13,14]. We found that a one-step replica symmetry broken solution described the glass phase $T < T_c$ with $C(x) \sim T_c \ln|x|$ at variance with the result of CO, while $C(x) \sim T \ln|x|$ for $T > T_c$. Such a discontinuity in the slope was observed in [21]. The simulation [20] was performed at such weak disorder that the system size was shorter than the length predicted by the GVM [13] beyond which glassy behavior can be observed in static quantities. One can argue that a variational method, such as the GVM, is approximate even in pure systems and misses some of the effects of nonlinearities better captured by the RG. On the other hand, this solution contains the feature of RSB and its compatibility with numerical calculations suggests that it has some relevance for the physics. Thus, it would be quite interesting to have a direct evidence that spontaneous RSB do occur in this model.

In this Letter we present such an evidence. We construct RG recursion relations, for the random field XY model, for couplings between replicas of arbitrary symmetry. It contains the CO recursion relations as a particular case. It allows for a more general way of taking the limit $n \to 0$, using Parisi-type matrices, while remaining in the perturbative regime of the RG. The new operators introduced in the limit $n \to 0$ are marginally relevant at $T = T_c$ and thus can be incorporated in the framework of RG. We show that the low-temperature phase does exhibit spontaneous RSB. In the high-temperature phase where the RG flows to weak coupling and is therefore certainly correct, we compute the linear response to a small RSB perturbation. The associated susceptibility diverges at the transition as $\chi \propto (T - T_c)^{-\gamma}$, when $T \to T_c^+$. For $T < T_c$ the replica symmetric flow and the CO fixed point are unstable to a small RSB. This is to our knowledge the first physical model where this effect can be demonstrated in a controlled way.

The Hamiltonian of the 2D vortex-free XY model in a field of random amplitude and direction reads

$$H = \int d^2x \, \frac{c}{2}[\nabla\psi(x)]^2 - \eta(x)\nabla\psi(x) - \zeta_1(x)\cos[\psi(x)] - \zeta_2(x)\sin[\psi(x)], \qquad (1)$$

where $\overline{\eta_i(x)\eta_j(x')} = \Delta_0\delta_{ij}\delta(x - x')$ and $\overline{\zeta_i(x)\zeta_j(x')} = \Delta\delta_{ij}\delta(x - x')$ are two Gaussian white noises. Equation (1) also describes flux lines with displacements u and average spacing a in $d = 1 + 1$ dimensions, with $\psi = 2\pi u/a$. Δ is the amplitude of disorder with Fourier component close to $2\pi/a$ and Δ_0 is the long wavelength disorder. Δ_0 is generated if not present originally in the model.

After replication of Eq. (1) and averaging one obtains

$$\frac{H_{\mathrm{eff}}}{T} = \int d^2x \sum_{ab} \left[\frac{K_{ab}^{-1}}{2\pi} \nabla\phi^a \nabla\phi^b - \frac{g_{ab}}{(2\pi)^2} \cos(2[\phi^a(x) - \phi^b(x)]) \right], \qquad (2)$$

where we set $g_{aa} = 0$. We have used for convenience

$$K_{ab}^{-1} = \frac{4\pi c}{T}\delta_{ab} - \frac{4\pi\Delta_0}{T^2}, \qquad \frac{g_{ab}}{(2\pi)^2} = \frac{\Delta}{2T^2}, \qquad (3)$$

and $\phi^a = \psi^a/2$. This defines the "bare" or starting value parameters, which of course are replica symmetric. We now consider a more general situation where the "renormalized" parameters K_{ab} and g_{ab} have arbitrary symmetry with respect to the group of replica permutation. We use the parametrization

$$K_{ab}^{-1} = \delta_{ab} - k_{ab}, \qquad (4)$$

and define the connected part $k_c = \sum_b k_{ab}$. As will be obvious later, there is a transition at temperature $T_c = 4\pi c$ in the model (1). Using Eqs. (3) and (4), this corresponds to $k_c = 0$, and one has more generally $k_c = (T - T_c)/T = \tau$, where τ is the reduced temperature.

In previous applications of RG to disordered systems, recursion relations are established for an arbitrary number n of replica. Then replica symmetry is assumed and the RG equations become simple functions of n. The continuation $n \to 0$ is then easily taken. In that respect the use of replica is a trick of graph counting. One can generally establish identical RG equations directly by considering disorder propagators, a method which we call "replica symmetric perturbation theory." However, in a glassy system, where many metastable states exist, one can question the validity of such a RG procedure. A crucial assumption of the RG is that one can simply integrate over the short scale degrees of freedom independently of the larger scales to produce an equivalent renormalized Hamiltonian. Such a separation between scales is far from obvious in glassy systems. Short scale degrees of freedom may well be determined by the local minimum they belong to, thus depend in effect on larger scales. The presence of long-range effects is supported by the massless modes found in the expansions around Parisi's solution [6]. However, the replicated Hamiltonian is translationally invariant and all these problems are buried in the proper taking of the limit $n \to 0$. In the case where the standard replica symmetric limit $n \to 0$ does not work, a Parisi-type RSB might allow one to construct a correct RG for glassy systems. We will therefore construct RG equations keeping the full matrix structure of the couplings. Since we are looking for a scale invariant theory *near the transition,* we consider perturbation theory in τ, in the small matrix parameters $2k_{ab} - k_{aa} - k_{bb}$, and in g_{ab}. This leaves unconstrained the degree of freedom of a constant shift of all the elements of k_{ab}. K_{ab}^{-1} is easily inverted as $K_{ab} = (1 - k_c)^{-2}[(1 -$

$2k_c)\delta_{ab} + k_{ab}] \simeq \delta_{ab} + k_{ab}$, which is valid even if the k_{ab} are large provided the above-mentioned parameters are small.

The constants g_{ab}, k_{ab}, and $k_c = \tau$ correspond to quadratic interactions in an equivalent fermion problem [22]. Using standard RG methods, either on the fermion form [22] or on the Hamiltonian (2), one can derive to second order the general RG for the replicated system

$$\frac{dg_{ab}}{dl} = (2k_{ab} - k_{aa} - k_{bb})g_{ab} + \frac{1}{2}\sum_{c\neq a,b} g_{ac}g_{bc}\,,$$

$$\frac{dk_{ab}}{dl} = \frac{1}{4}g_{ab}^2, \qquad \frac{d\tau}{dl} = 0\,, \tag{5}$$

where l is the standard logarithmic scale. In these equations $a \neq b$ is implied. The reduced temperature τ is unrenormalized. The subspace of replica symmetric parameters $g_{ab} = g$, $k_{a\neq b} = k$ is preserved by the RG and Eqs. (5) for g, k, and τ reduce to the CO equations in the limit $n = 0$: $dg/dl = -2\tau g - g^2$, $dk/dl = \frac{1}{4}g^2$. In the high-temperature phase, $\tau > 0$, the disorder g is irrelevant and the fixed point is a pure Gaussian system with $C(x) = \overline{\langle\phi(x) - \phi(0)\rangle^2} \sim [T/T_c + O(g)]\ln|x|$. The renormalization of the off-diagonal part k contributes to $O(g)$, since k goes to a constant (which stays finite up to $\tau = 0$). In the low-temperature phase, there is a fixed point at $g^* = 2|\tau|$. However, k flows to infinity. This is a peculiar situation which within the replica symmetric scheme does not lead to inconsistencies, since k does not feedback to any order in perturbation [only averages $\overline{(\sum_a C_a\phi_a)^2}$ with $\sum_a C_a = 0$ appear]. The flow of k was used [17] to predict $C(x) \sim B(\ln|x|)^2$.

We now look at the RG for coupling constants parametrized by Parisi matrices. g_{ab} and k_{ab}, $a \neq b$, become the functions $g(u)$, $k(u)$, with $0 < u < 1$. The symmetric case corresponds to constant functions. The independent variables are now τ, $g(u)$, and $k(u)$. The equations read

$$\frac{dg(u)}{dl} = \{-2\tau + 2[k(u) - \langle k\rangle]\}g(u) - \langle g\rangle g(u) - \frac{1}{2}\int_0^u dv[g(v) - g(u)]^2, \qquad \frac{dk(u)}{dl} = \frac{1}{4}g(u)^2, \tag{6}$$

where $\langle k\rangle = \int_0^1 dv\, k(v)$. These functions depend on l but this will be written explicitly only when needed. There is now a spectrum of dimensions, given by $-\tau + k(u) - \langle k\rangle$, for the operators corresponding to the nonreplica symmetric couplings g_{ab}. The dimension of these operators should be small to neglect the effect of higher order replica terms. Also $k(u)$ now feeds back in the equation for $g(u)$, thus one anticipates an instability.

We first check the stability of the replica symmetric flow in the low-temperature phase. To separate the replica symmetric part, we write $g(u) = g_l + \epsilon(u)$ and $k(u) = k_l + A(u)$ with $\langle\epsilon\rangle = 0$ and $\langle A\rangle = 0$. At the CO fixed point $g = g^*$, k_l being the running coupling constant when $g = g^*$, one obtains from Eq. (6) the deviations from the replica symmetric solution to linear order:

$$\frac{d\epsilon(u)}{dl} = 2g^*A(u), \qquad \frac{dA(u)}{dl} = \frac{g^*}{2}\epsilon(u)\,. \tag{7}$$

Thus the RG flow is clearly *unstable* to replica symmetry breaking when $\tau < 0$. The eigenvalue of instability is $\lambda = 2|\tau|$. When $\tau < 0$, there is no small coupling fixed point and the flow goes to strong coupling [23]. This is clear from Eqs. (6), since $A(u)$ can only reach a fixed point if $g^2(u) - \langle g^2\rangle$ goes to zero. This is a strong indication that the low-temperature phase corresponds to a replica symmetry broken solution. To conclude unambiguously on that issue, we now consider the high-temperature phase $\tau > 0$, where the RG is exact since there is a weak coupling fixed point $g(u) = 0$ [and $A(u)$ small]. By computing the linear susceptibility to small RSB perturbation we show that spontaneous RSB occurs at T_c.

Let us define the susceptibility to RSB as follows. We start for $\tau > 0$ with a given disorder g_0 and add a small RSB perturbation $g(u) = g_0 + \epsilon_0(u)$ uniform in space. This will result in a replica nonsymmetric part in the correlation functions $\langle\phi_a(q)\phi_b(-q)\rangle = (\delta_{ab} + k_{ab}^*)/q^2$, where k_{ab}^* is the renormalized coupling at the fixed point $g(u) = 0$. A susceptibility is defined as

$$A^*(u) \equiv k^*(u) - \langle k^*\rangle = \chi\epsilon_0(u)\,, \tag{8}$$

in the limit $\epsilon_0 \to 0$. χ should be a function of u but it turns out that it has the above simple form. Alternatively, one can add a quadratic RSB perturbation $\delta k_{ab}\nabla\phi_a\nabla\phi_b$ and define a corresponding susceptibility χ' as in Eq. (8). To compute χ let us expand the RG equations (6) to linear order in ϵ_0. As can be seen from Eqs. (6), A can be considered as linear in ϵ_0:

$$\frac{dg}{dl} = -2\tau g - g^2, \qquad \frac{dA(u)}{dl} = \frac{1}{2}g\epsilon(u)\,,$$

$$\frac{d\epsilon(u)}{dl} = -(2\tau + g)\epsilon(u) + 2gA(u)\,. \tag{9}$$

At linear order, each u can be treated independently. We perform the following rescalings: $x = 2\tau l$, $\epsilon(u) = \epsilon_0(u)f$, $A(u) = \epsilon_0(u)a$, and $g = 2\tau\tilde{g}$. The first equation of (9) integrates to give $\tilde{g}(x) = e^{-x}/(\alpha + 1 - e^{-x})$, where we have defined $\alpha = 2\tau/g_0$. Defining $Z(x) = (2/\alpha)\int_0^x dy\, A(y) + 1$, Eqs. (9) give $f(x) = \alpha e^{-x}Z(x)/(\alpha + 1 - e^{-x})$, where $Z(x)$ satisfies

$$\frac{d^2Z}{dx^2} = Z(x)e^{-2x}/(\alpha + 1 - e^{-x})^2 \tag{10}$$

with initial conditions $Z(0) = 1$ and $Z'(0) = 0$. The problem depends thus only on one variable $\alpha = 2\tau/g_0$. The susceptibility is obtained from the asymptotic value A_∞ and is $\chi = \lim_{x\to\infty}\alpha Z'(x)/2$.

A full solution of Eq. (10) can be obtained numerically, but χ can be estimated from considering the two asymptotic regimes $\alpha \gg 1$ and $\alpha \ll 1$. In the first regime $\alpha \gg 1$, Eq. (10) becomes $Z''(x) = e^{-2x}Z(x)/\alpha^2$ solved in terms of Bessel functions $Z(x) = c_1 I_0(e^{-x}/\alpha) + c_2 K_0(e^{-x}/\alpha)$. $c_{1,2}$ are two constants determined from the initial conditions. This gives the susceptibility in that regime as $\chi = \frac{1}{2}I_1(g_0/2\tau) \sim g_0/8\tau$. To investigate the behavior close to the transition one has to look at $\alpha \ll 1$ and to match two regimes in x. For $x \ll 1$, Eq. (10) can be written as $Z''(x) = Z(x)/(\alpha + x)^2$, whose solution is a power law $Z \propto (\alpha + x)^{-\nu}$, where ν satisfies the golden mean equation $\nu^2 + \nu = 1$. We define $\gamma = (\sqrt{5} - 1)/2 \simeq 0.618$. The solution is $Z(x) = [(1 + x/\alpha)^{-\gamma} + \gamma^2(1 + x/\alpha)^{1/\gamma}]/(1 + \gamma^2)$. When x becomes of order 1, one can use the equation of the first regime with $\alpha \rightarrow \alpha + 1$. A simple matching at $x \approx 1$ gives

$$\chi \propto C(g_0/2\tau)^{\gamma}. \quad (11)$$

We performed a numerical integration of Eq. (10) which confirms the analytic estimates, Eq. (11), and gives $C \approx 0.165$. χ' can be computed similarly from Eq. (10), with modified initial conditions $Z(0) = 0$ and $Z'(0) = 2/\alpha$ and has the same divergence as χ. There is also a nonlinear response in ϵ_0 in the high-temperature phase. A runaway flow occurs for smaller and smaller values of ϵ_0 when $\tau \rightarrow 0$, roughly when $\chi\epsilon_0 > \tau$.

The divergence of χ in Eq. (11) when $\tau \rightarrow 0$ for fixed g_0 shows that RSB occurs spontaneously [24] for $\tau \leq 0$. Quantities like $\langle\phi_a(q)\phi_b(-q)\rangle$ acquire a replica nonsymmetric part. Defining $q_{ab} = \langle\nabla\phi_a(x)\nabla\phi_b(x)\rangle$, a possible order parameter for RSB is $Q_{ab} = 2q_{ab} + \sum_{c\neq a} q_{ac} + \sum_{c\neq b} q_{bc}$. The perturbation δk_{ab} is the conjugated field to Q_{ab}. Thus χ' is the inverse effective mass of Q_{ab} whose divergence signals an instability *à la* de Almeida and Thouless [25] when $T \leq T_c$. By analogy with the sine-Gordon model, a reasonable interpretation of the runaway flow of g_{ab} is that a "mass" develops for some modes below T_c. This scenario is in good agreement with the GVM [13,14], where correlation functions become nonreplica symmetric. A fraction $1 - u_c$ of the modes become massive below T_c, with $u_c \approx T/T_c$. These modes correspond, in the Coulomb gas, to types of charges which unbind, a fraction u_c remaining massless (corresponding to bound types of charges). The GVM, being nonperturbative contrary to RG, allows for the generation of a mass.

In this Letter, we have shown that replica symmetry breaking can be incorporated in the RG. We found new relevant operators which break replica symmetry when $n \rightarrow 0$ for $T \leq T_c$. The associated susceptibility diverges in the high-temperature phase $T > T_c$. The replica symmetric fixed point found by Cardy and Ostlund is unstable. RSB is likely to have observable consequences in the dynamics, such as breaking of the fluctuation dissipation theorem, aging, and persistent correlations, as in [26]. It is also probable that the static correlation functions should be different from the one predicted by the symmetric RG, as hinted at by the variational method. Further numerical and experimental results would be of great interest.

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Note added.—After submission, we received a preprint by J. Kierfeld, where he independently considered the instability of the CO fixed point to one-step RSB.

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