Geometric Properties of the Chaotic Saddle Responsible for Supertransients in Spatiotemporal Chaotic Systems

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Superlong chaotic transients have been observed commonly in spatiotemporal chaotic dynamical systems. The phenomenology is that trajectories starting from random initial conditions behave chaotically for an extremely long time before settling into a final nonchaotic attractor. We demonstrate that supertransients are due to nonattracting chaotic saddles whose stable manifold measures have fractal dimensions that are arbitrarily close to the phase-space dimension. Numerical examples using coupled map lattices are given.

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Superlong chaotic transients (referred to as "supertransients") occur commonly in spatiotemporal chaotic dynamical systems [1,2]. In such a case, trajectories starting from random initial conditions wander chaotically for an arbitrarily long time before settling into a final attractor which is usually nonchaotic (periodic or quasiperiodic). Crutchfield and Kaneko first observed in numerical experiments that spatially extended systems exhibit chaotic transients; transients long enough so that the observation of the system's asymptotic attractor is practically impossible [1]. More recently, Hastings and Higgins showed the existence of complex transient dynamics in simple discrete-time, spatially extended ecological models for a species with alternating reproduction and dispersal [2]. They observed that with sufficiently strong nonlinearity, the time required for the system to settle into the asymptotic attractor is usually very long, approaching thousands of generations. These results are consistent with observed behavior in populations of certain biological species [2].

Previous studies have established that transient chaos is due to nonattracting chaotic saddles in the phase space [3–5]. When there is a chaotic saddle in the phase space, trajectories originating from random initial conditions usually wander in the vicinity of the chaotic saddle for a finite amount of time before escaping the chaotic saddle and settling into the final attractor. In this paper, we investigate the geometric properties of the chaotic saddle which is responsible for the supertransients in spatiotemporal chaotic systems. It is found that the natural measure of the stable manifold of the chaotic saddle, defined approximately as the set of initial conditions whose trajectories wander for an arbitrarily long time on the chaotic saddle, possesses a fractal dimension which is arbitrarily close to the phase-space dimension. As a consequence, the average lifetime of the chaotic transient induced by the chaotic saddle is arbitrarily large.

We consider a spatiotemporal system for which there is a nonattracting chaotic saddle Λ and a nonchaotic attractor A in the phase space. All initial conditions, except a set of measure zero, asymptote eventually to A. Trajectories starting from random initial conditions typically wander chaotically near the chaotic saddle Λ for a finite time before settling into A. For different initial conditions, lengths of the chaotic transient are different. An average lifetime can be defined as follows. Suppose at $t = 0$ we choose N_0 initial conditions, where N_0 is large. Evolve these N_0 initial conditions under the dynamics. Let $N(t)$ be the number of trajectories that have not settled into A at time t. Then, due to the chaotic nature of Λ, $N(t)$ typically decays exponentially with time [3],

$$N(t) = N_0 \exp(-t/\tau), \tag{1}$$

where τ is the average lifetime of the chaotic transient. For two-dimensional maps, τ can be related to the fractal dimension of the stable manifold measure of the chaotic saddle Λ as follows [5],

$$\tau \approx 1/(1 - d_s)\lambda_1, \tag{2}$$

where d_s is the fractal dimension of the set of intersecting points of a one-dimensional line with the stable manifold of the chaotic set, and λ_1 is the maximum Lyapunov exponent for trajectories on the chaotic saddle. In the same spirit, we can argue the same relation for spatiotemporal chaotic systems [6]. While such an argument is not rigorous, the key implication is that the average transient time can be arbitrarily large if d_s is arbitrarily close to 1. Note that the dimension of the stable manifold measure of the chaotic saddle is $N - 1 + d_s$, where N is the phase-space dimension. Thus, Eq. (2) suggests that supertransients in spatiotemporal chaotic systems are due to chaotic saddles whose stable manifold measures have fractal dimensions that are arbitrarily close to the phase-space dimension [7].

Our numerical example is the diffusively coupled logistic map lattice [8],

$$x_{n+1}^i = (1 - \delta)f(x_n^i) + \frac{\delta}{2}[f(x_n^{i+1}) + f(x_n^{i-1})], \quad i = 1, \ldots, N, \tag{3}$$

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where $f(x) = ax(1 - x)$ is the one-dimensional logistic equation [9], i and n denote discrete spatial site and time, N is the number of coupled maps, and δ is the coupled strength; the coupling exists only among nearest neighbors (diffusively coupling). Periodic boundary conditions, i.e., $x^1 = x^{N+1}$, is assumed. Equation (3) was first proposed by Kaneko [8] as a simple model for investigating the phenomenology of spatiotemporal chaos. It is perhaps the most extensively studied model spatiotemporal dynamical system in the literature for the past decade. We choose the following parameter values: $a = 4$, $\delta = 0.8$, and $N = 20$. Figure 1 shows a typical time series obtained at site 8 resulting from a random initial condition. The trajectory behaves chaotically for a very long time (about 10^5 iterations) before settling into the final attractor.

To determine the average transient lifetime τ, we compute snapshots of histograms of λ_1 for a large number of uniformly chosen initial conditions. Specifically, a 32×32 grid of initial conditions was chosen in the two-dimensional region defined by $0 \leq x(8) \leq 1$ and $0 \leq x(9) \leq 1$, while values of $x(j)$ ($j = 1, \ldots, N$, $j \neq 8, 9$) of these initial conditions are fixed. Values of λ_1 for these 1024 initial conditions were then computed after an initial transient of 10 000 iterations [10]. Histograms of λ_1 at time steps $t_n = 10\,000n$ ($n = 1, 2, \ldots, 20$) were constructed. Figure 2(a) shows such a histogram at $t_1 = 10\,000$. There are two peaks: one at $\lambda_1 = 0$ and another at $\lambda_1 \approx 0.36$. As time evolves, the height of the peak at $\lambda_1 \approx 0.36$ decreases, indicating that this peak corresponds to a nonattracting chaotic saddle. The peak at $\lambda_1 = 0$ represents a quasiperiodic attractor. As time progresses, trajectories escape the chaotic saddle and approach asymptotically the quasisperiodic attractor. It is found that the extent of the quasiperiodic attractor in phase space is quite small. Thus the value $x(8)$ in the time series on the quasiperiodic attractor appears to be constant, as shown in Fig. 1. Figure 2(b) shows the number of chaotic trajectories $N(t)$ vs t in a semilogarithmic plot, where a trajectory is counted as chaotic at time t if $\lambda_1 > 0.3$ at t. The plot can be fitted by a straight line, indicating that the decay of chaotic trajectories is exponential. The slope of the fitted line is $-1.93 \times 10^{-5} \pm 8.47 \times 10^{-7}$, which gives $\tau \approx 5.18 \times 10^4$, a very long transient.

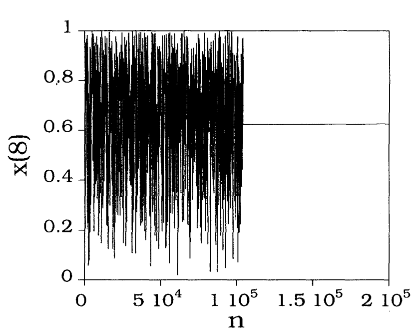

FIG. 1. A time series from a random initial condition for the diffusively coupled logistic map Eq. (4) for $N = 20$, $a = 4$, and $\delta = 0.8$. The trajectory behaves chaotically for an extremely long time (over 10^5 iterations) before settling into the final attractor. Such superlong chaotic transients are typical in spatiotemporal dynamical systems [1,2].

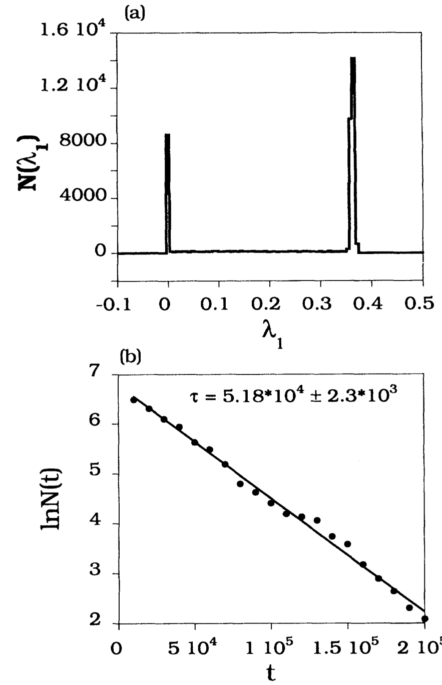

FIG. 2. (a) A histogram of the maximum Lyapunov exponent λ_1 computed using a uniform grid of 32×32 initial conditions chosen on a two-dimensional section [$x(8)$ and $x(9)$] of the 20-dimensional phase space. There are two peaks, one at $\lambda_1 \approx 0.36$ and the other at $\lambda_1 = 0$. The former corresponds to a chaotic transient and the latter represents a quasiperiodic attractor. (b) The plot of $N(t)$, number of chaotic trajectories with $\lambda_1 > 0.3$ at time t, on a semilogarithmic plot. The decay of $N(t)$ is exponential. The average transient lifetime is $5.18 \times 10^4 \pm 2.3 \times 10^3$.

To compute the stable manifold of the chaotic saddle, we use the "sprinkle method" [5,11] by which the stable manifold is approximated by the set of initial conditions that still remain chaotic at time t_c, where t_c is large. Figure 3(a) shows the set of initial conditions (black dots) drawn from a 200×200 grid on the $x(8)$-$x(9)$ plane whose trajectories have $\lambda_1 > 0.3$ at $t_c = 20\,000$. The picture thus represents a two-dimensional cross section of the stable manifold in the 20-dimensional phase space. The intermingling appearance of the stable manifold suggests that its dimension may be arbitrarily close to the dimension of the entire phase space.

The fractal dimension of the stable manifold measure can be computed using the uncertainty algorithm introduced by Grebogi *et al.* [12] to calculate the dimension of fractal basin boundaries for dynamical systems with multiple attractors. The procedure is as follows: Randomly choose an initial condition x_0 on an arbitrary one-dimensional line. Define $x_0' = x_0 + \epsilon$, where ϵ is a small perturbation. Determine whether the values of λ_1, computed for these two initial conditions at

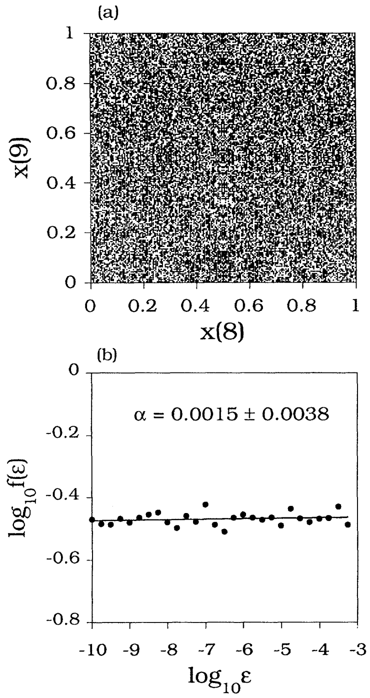

FIG. 3. (a) The stable manifold of the chaotic saddle (black dots) on the two-dimensional cross section $x(8)$ and $x(9)$. (b) Plot of the fraction of uncertain initial conditions $f(\epsilon)$ versus the uncertainty ϵ on a base-10 logarithmic scale. The uncertainty exponent is estimated to be 0.0015 ± 0.0038 and, hence, the fractal dimension d_s is 0.9985 ± 0.0038, a quantity that cannot be distinguished from 1. This leads to arbitrarily long chaotic transients.

$t = 20\,000$, are distinct; i.e., one ≈ 0 and another >0.3. If yes, then the initial condition x_0 is called uncertain. For a given perturbation ϵ, a fraction of uncertain initial conditions $f(\epsilon)$ can be computed by randomly choosing many initial conditions and determining if they are uncertain. For fractal sets, $f(\epsilon)$ decreases with decreasing ϵ, typically scaling with ϵ as $f(\epsilon) \sim \epsilon^{\alpha}$, where α is the uncertainty exponent [12]. The fractal dimension of the set of intersecting points of the stable manifold with the one-dimensional line is given by $d_s = 1 - \alpha$ (the so-called uncertainty dimension) [12,13]. Figure 3(b) shows $\log_{10} f(\epsilon)$ vs $\log_{10}\epsilon$, where the initial conditions were drawn on the line defined by $x(9) = 0.8$ in Fig. 3(a). The uncertainty exponent is estimated to be $\alpha = 0.015 \pm 0.038$ and, hence, $d_s = 0.985 \pm 0.038$, a value which cannot be distinguished from 1 [14]. This indicates that τ can be arbitrarily large, inconsistent with the result of very long transient observed in Fig. 2(b).

To illustrate that $d_s \approx 1$ and supertransients are common for the coupled logistic map lattice [Eq. (3)], we have explored the two-dimensional parameter space (a and δ) of Eq. (3). For a given parameter pair, the maximum Lyapunov exponent computed at finite time depends extremely sensitively on the initial condition as a consequence of the near-zero uncertainty exponent. This fact provides a way to detect supertransients in parameter space; we simply compute λ_1 at finite time for many parameter values using a fixed initial condition. Plot the parameter pairs which lead to a nonchaotic trajectory within this time. Parameter regions where supertransients occur will exhibit similar "riddled" structures as in Fig. 3(a). We find that there are substantial parameter regions of Eq. (3) that exhibit supertransients [15].

To illustrate that supertransients and $d_s \approx 1$ are not unique to the diffusively coupled logistic map lattice [Eq. (3)], we have also investigated the following globally coupled Zaslavsky map [16] lattice,

$$\begin{aligned} x_{n+1}^i &= \left[x_n^i + \frac{1 - e^{-a}}{a} y_n^i \right] \mathrm{mod}(2\pi), \\ y_{n+1}^i &= e^{-a} y_n^i + k \sin\left[(1-\delta) x_{n+1}^i + \frac{\delta}{N-1} \sum_{j, j\neq i}^{N} x_{n+1}^j \right], \end{aligned} \tag{4}$$

where a and k are parameters of the single Zaslavsky map, and δ is the coupling strength. We have found that it is also common for this system to exhibit supertransients. For instance, at $k = 8$, $a = 0.5$, $N = 10$, and $\delta = 0.2$, we have $\tau = 1.44 \times 10^5 \pm 6.4 \times 10^3$ and $d_s = 0.999\,83 \pm 0.000\,34$. Examination of parameter space also reveals features similar to those in the coupled logistic map lattice [Eq. (3)].

We remark that supertransients can also occur in low-dimensional chaotic systems. For example, in the event of boundary crisis [3] where a chaotic attractor is suddenly destroyed and is converted into a nonattracting chaotic saddle as a system parameter p passes through a critical value p_c, supertransients occur when p is immediately above p_c. Nonetheless, the average transient lifetime τ decreases algebraically as p increases above p_c. In this sense, supertransients, meaning that τ is arbitrarily large, only occur in an arbitrarily small parameter interval in the vicinity above p_c. In contrast, in spatiotemporal chaotic systems supertransients occur in substantial portions of the parameter space [15]. Therefore, we expect supertransients to be common in spatiotemporal chaotic systems.

The main contribution of this Letter is extensive numerical computations demonstrating that supertransients in coupled map lattice systems are associated with chaotic saddles whose stable manifold measures have fractal dimensions arbitrarily close to the phase-space dimension. The theoretical relation [Eq. (2)], which provides a base and a guide for our numerical experiments, has been established previously for low-dimensional chaotic systems [5]. It should be noted, however, that a more general conjecture which relates the transient lifetime to geometrical properties of the chaotic saddles has been proposed by Kantz and Grassberger [11]. In their conjecture, the chaotic transient lifetime for maps of any dimension is expressed by

$$\tau = 1 \Big/ \sum_i (1 - D_1^i)\lambda_i, \tag{5}$$

where D_1^i and λ_i are the partial information dimension and Lyapunov exponent along the unstable directions, respectively. Kantz and Grassberger proposed that if $D_1^i \approx 1$ for a particular unstable direction, escape of trajectories does not occur or is very slow along this direction. Escape within short time (or short chaotic transient) occurs only along these unstable directions with $D_i < 1$. For a chaotic system, the most unstable direction is the direction associated with the maximum Lyapunov exponent λ_1. Escape is thus most likely to occur along this direction. Note that $D_1^1 \approx d_s$ [11]. In coupled map lattices which exhibit supertransients, our numerical experiments reveal that d_s, or D_1^1, is arbitrarily close to 1. This implies that D_1^i $(i > 1)$ will be even closer to 1 because escape is less probable along the directions with $i > 1$ and, hence, the leading contribution to τ comes from $(1 - D_1^1)\lambda_1$. Therefore, for coupled map lattices, the Kantz-Grassberger conjecture [Eq. (5)] is equivalent to Eq. (2). We stress, however, that this equivalence holds only for systems with $d_s \approx 1$, which indicates supertransients. The coupled map lattice systems we investigate in this Letter appear to fall within this category.

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[6] Consider a one-dimensional line intersecting the stable manifold of the chaotic set Λ. Here we assume that the dimension of the stable manifold measure of Λ is sufficiently large so that a randomly chosen line has a nonzero probability to intersect Λ. Generally, intersecting points of Λ with the line form a Cantor set [3] with a box-counting dimension d_s [J.D. Farmer, E. Ott, and J.A. Yorke, Physica (Amsterdam) **7D**, 153 (1983)]. Let $\lambda_1 > 0$ be the maximum Lyapunov exponent of a dense trajectory on Λ. Consider intervals on the line with width $\epsilon \sim e^{-\lambda_1 t}$, where t is large [there are approximately $N(\epsilon) \equiv \epsilon^{-d_s}$ such intervals] and uniformly distribute initial conditions on the one-dimensional line. The total length of all these intervals is rougly $\sim \epsilon N(\epsilon) \sim \epsilon^{1-d_s}$. Then the fraction of initial conditions that fall into these small intervals is $\sim \epsilon^{1-d_s}$. At time t the lengths of these intervals will be $\sim \epsilon e^{\lambda_1 t} \sim 1$ and, hence, most initial conditions chosen in these intervals will leave the chaotic saddle Λ for time greater than t. There are roughly $\sim \epsilon^{1-d_s}$ such trajectories which are chaotic for time t. Therefore, we arrive at $\exp[-\lambda_1(1 - d_s)t] \sim \exp[-t/\tau]$, which gives Eq. (2). Notice that there are usually many positive Lyapunov exponents for spatiotemporal systems. However, at large t only λ_1 is responsible for the exponential stretch in length.

[7] A characteristic of supertransients in spatiotemporal systems is that the transient lifetime increases exponentially as the system size increases [1]. This is not reflected in Eq. (2). In fact, as τ gets large, d_s will be very close to 1, and it is impractical to distinguish numerically the closeness of d_s to 1 as τ increases further. Therefore, the conclusion of this paper provides no check of the dependence of the transient lifetime on system size. Nonetheless, the major point of the paper is that the chaotic saddle, which is responsible for supertransients, possesses the geometric property that d_s is arbitrarily close to one. At present, to our knowledge, why transient lifetime increases exponentially with system size remains unknown.

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