

## Critical Behavior in Gravitational Collapse of Radiation Fluid: A Renormalization Group (Linear Perturbation) Analysis

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A scenario is presented, based on renormalization group (linear perturbation) ideas, which can explain the universality and scaling observed in a numerical study of gravitational collapse of radiation fluid. In particular, it is shown that the critical exponent  $\beta$  and the largest Lyapunov exponent  $\text{Re}\kappa$  of the perturbation is related by  $\beta = (\text{Re}\kappa)^{-1}$ . We find the relevant perturbation mode numerically, and obtain a fairly accurate value of the critical exponent  $\beta \approx 0.355\,801\,92$ , also in agreement with that obtained in numerical simulation.

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*1. Introduction.*—Gravitational collapse with the formation of black holes is one of the main problems of classical general relativity. Recently, Choptuik [1] discovered a “critical behavior” of the gravitational collapse of a massless scalar field by a numerical study. His result can be summarized as follows: Let the initial distribution of the scalar field be parametrized smoothly by a parameter  $p$ , such that the solutions with the initial data  $p > p^*$  contain a black hole, while those with  $p < p^*$  do not. For several one parameter families investigated, near critical solutions ( $p \approx p^*$ ) satisfy the following: (i) the critical solution (i.e.,  $p = p^*$ ) is *universal* in the sense that it approaches the identical spacetime for all families, (ii) the critical solution has a discrete self-similarity, and (iii) for supercritical solutions ( $p > p^*$ ) the black hole mass satisfies  $M_{\text{BH}} \propto (p - p^*)^\beta$  and the critical exponent  $\beta$ , which is about 0.37, is universal for all families. Abrahams and Evans [2] found similar phenomena in the axisymmetric collapse of a gravitational wave.

Evans and Coleman [3] found similar phenomena with  $\beta \approx 0.36$  in a spherically symmetric collapse of radiation fluid, in which case the self-similarity is not discrete but *continuous*. Employing a self-similar ansatz they also found a numerical solution which fits the inner region of the near-critical solutions very well, and suggested that linear analysis around the solution will be useful.

In this Letter, we present a scenario that explains the critical behavior observed in the radiation fluid collapse. We directly relate eigenvalues (Lyapunov exponents) of perturbations of the self-similar solution to the critical exponent  $\beta$ , using an argument of renormalization group transformation. The formulation is general, and could be applied to other models with approximate self-similarity. We find the eigenvalues of the perturbation by numerical analysis, and find that the value of the exponent  $\beta$  predicted from our analysis matches very well with that observed in [3].

We present our picture in Sec. 2. After reviewing the equations of motion in Sec. 3, and the self-similar solution in Sec. 4, we confirm our picture to an extent in Sec. 5 by a numerical study. Section 6 is for conclusions and discussions.

*2. A scenario based on renormalization group ideas.*—We give a formalism for linear perturbations around a self-similar spacetime from the point of view of the renormalization group, which proved to be extremely successful in the study of critical phenomena [4]. We have benefited from the formulation of [5]. The argument is general, but the notation is so chosen as to pave the shortest path to our analysis on radiation fluid. We introduce the “scaling variable”  $s$  and the “spatial coordinate”  $x$ , which are related to the time  $t$  and the radial coordinate  $r$  by  $s \equiv -\ln(-t)$ ,  $x \equiv \ln(-r/t)$ .

*2(a). Renormalization group and linear perturbations.*—Let  $h = (h_1, h_2, \dots, h_m)$  be functions of  $s$  and  $x$  which satisfy a partial differential equation (PDE)

$$L\left(h, \frac{\partial h}{\partial s}, \frac{\partial h}{\partial x}\right) = 0. \quad (1)$$

Suppose that the PDE is invariant under the “scaling transformation” (translation in  $s$ ) with  $s_0 \in \mathbf{R}$

$$h(s, x) \mapsto h(s + s_0, x). \quad (2)$$

A renormalization group transformation (RGT)  $\hat{\mathcal{R}}_{s_0}$  is a transformation on the space of functions of  $x$ ,

$$\hat{\mathcal{R}}_{s_0} : H \mapsto H^{(s_0)}, \quad (3)$$

where

$$H(x) \equiv h(0, x), \quad (4)$$

$$H^{(s_0)}(x) = h(s_0, x). \quad (5)$$

Namely, one obtains  $H^{(s_0)}$  by evolving the initial data  $H$  at  $s = 0$  by the PDE to  $s = s_0$ .  $\hat{\mathcal{R}}_{s_0}$  forms a semigroup with parameter  $s_0$ , and we denote its generator by  $\widehat{D\mathcal{R}}$ , i.e.,  $\widehat{D\mathcal{R}} = \lim_{s_0 \rightarrow 0} (\hat{\mathcal{R}}_{s_0} - 1)/s_0$ . In this context, a self-similar solution  $h_{ss}(s, x) = H_{ss}(x)$  can be considered as a *fixed point* of  $\hat{\mathcal{R}}_{s_0}$  for any  $s_0$ , and is characterized by  $\hat{\mathcal{R}}_{s_0} H_{ss} = H_{ss}$  or  $\widehat{D\mathcal{R}} H_{ss} = 0$ .

The *tangent map* of  $\hat{\mathcal{R}}_s$  at a fixed point  $H_{ss}$  is defined as a transformation on functions of  $x$ :

$$\hat{T}_{s_0} F \equiv \lim_{\epsilon \rightarrow 0} \frac{\hat{\mathcal{R}}_{s_0}(H_{ss} + \epsilon F) - H_{ss}}{\epsilon}. \quad (6)$$

An *eigenmode*  $F(x)$  of  $\widehat{DT} = \lim_{s_0 \rightarrow 0} (\hat{T}_{s_0} - 1)/s_0$  is a function which satisfies ( $\kappa \in \mathbb{C}$ )

$$\widehat{DT} F = \kappa F. \quad (7)$$

These modes determine the flow of the RGT near the fixed point. A mode with  $\text{Re}\kappa > 0$ , a *relevant mode*, is a flow diverging from  $H_{ss}$ , and one with  $\text{Re}\kappa < 0$ , an *irrelevant mode*, is a flow converging to it. A  $\text{Re}\kappa = 0$  mode is called *marginal*.

2(b). *The critical solution.*—In this and the next subsection, we present a scenario which explains the observed critical behavior of radiation fluid, assuming that there is a *unique* relevant mode with eigenvalue  $\kappa$  (and, for simplicity, no marginal mode) around the fixed point  $H_{ss}$ . This assumption is confirmed to some extent in Secs. 3–5.

The assumption implies that the RG flow around the fixed point is shrinking, except for the direction of the relevant mode (Fig. 1). There will be a “critical surface” or a “stable manifold”  $S$  of the fixed point  $H_{ss}$ , of codimension one, whose points will all be driven towards  $H_{ss}$ . There will be an “unstable manifold”  $U$  of dimension one, whose points are all driven away from  $H_{ss}$ . A one parameter family of initial data  $I$  will in general intersect with the critical surface, and the intersection  $H_c$  will be driven to  $H_{ss}$  under the RGT:

$$\lim_{s \rightarrow \infty} |H_c^{(s)}(x) - H_{ss}(x)| = 0. \quad (8)$$

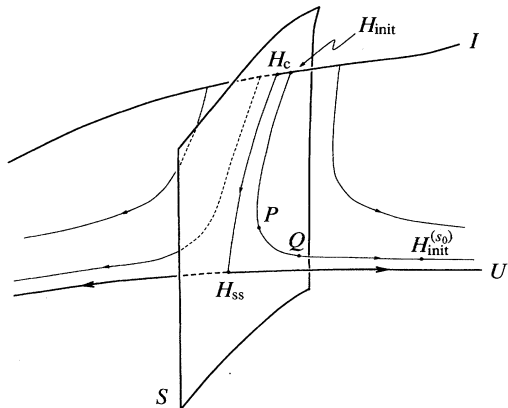


FIG. 1. Schematic view of RG flows near  $H_{ss}$ .

So,  $H_c$  are the initial data with critical parameter  $p^*$ . The existence of a critical solution for an arbitrarily chosen family of initial data thus supports the assumption of a unique relevant mode [6].

2(c). *The critical behavior.*—We now consider the fate of an initial data  $H_{init}$  in the one parameter family, which is close to  $H_c$  ( $\epsilon = p - p^*$ ):

$$H_{init}(x) = H_c(x) + \epsilon F(x). \quad (9)$$

We evolve these data to  $s = s_0$  (chosen later): It will first be driven towards  $H_{ss}$  along the critical surface, but eventually be driven away along the unstable manifold. Using linear perturbations we have [7]

$$H_{init}^{(s_0)} = \hat{\mathcal{R}}_{s_0} H_{init} = \hat{\mathcal{R}}_{s_0}(H_c + \epsilon F) \approx H_c^{(s_0)} + \epsilon \hat{T}_{s_0} F + O(\epsilon^2). \quad (10)$$

In the second term, only the relevant mode survives:

$$\hat{T}_{s_0} F = \exp(s_0 \widehat{DT}) F \approx e^{\kappa s_0} F_{rel}, \quad (11)$$

where  $F_{rel}$  is the component of the relevant mode in  $F$ . Because of (8), we finally have (for large  $s_0$  and  $x \lesssim s_0$ )

$$H_{init}^{(s_0)}(x) \approx H_{ss}(x) + \epsilon e^{\kappa s_0} F_{rel}(x). \quad (12)$$

Now we choose  $s_0$  so that the first and second terms in (12) become comparable, i.e.,

$$|\epsilon| e^{(\text{Re}\kappa)s_0} = O(1). \quad (13)$$

Now that the second term is of  $O(1)$ , the data  $H_{init}^{(s_0)}$  differ from  $H_{ss}$  so much that one can tell the fate of these data depending on the sign of  $\epsilon$ , and if a black hole is formed, the radius of its apparent horizon, and thus its mass, will be  $O(1)$  measured in  $x$ .

Finally, we translate the above result back into our original coordinate  $(t, r)$ . The relation  $r = e^{x-s}$  implies that the radius of the apparent horizon, which is  $O(1)$  measured in  $x$ , is in fact  $O(e^{-s_0})$  measured in  $r$ . So we have from (13)

$$M_{BH} \approx O(e^{-s_0}) \approx O(|\epsilon|^{1/(\text{Re}\kappa)}). \quad (14)$$

Therefore the critical exponent is given *exactly* [7] by

$$\beta = \frac{1}{\text{Re}\kappa}. \quad (15)$$

3. *Equations of motion.*—The line element of any spherically symmetric spacetime is written as

$$ds^2 = -\alpha^2(t, r) dt^2 + a^2(t, r) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (16)$$

The only coordinate transformation which preserves the form of Eq. (16) is

$$t \mapsto F^{-1}(t). \quad (17)$$

We assume the matter content is a perfect fluid having energy-momentum tensor  $T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b)$ , where  $u^a$  is a unit timelike vector and  $p = (\gamma - 1)\rho$ . We only consider the radiation fluid  $\gamma = 4/3$  in this Letter. The components of  $u^a$  can be written as  $u_t = \alpha(1 - V^2)^{-1/2}$  and  $u_r = \alpha V(1 - V^2)^{-1/2}$ , where  $V$  is the three-velocity of fluid particles. In terms of variables  $s \equiv -\ln(-t)$ ,  $x \equiv \ln(-r/t)$ , and introducing  $N \equiv \alpha a^{-1} e^{-x}$ ,  $A \equiv a^2$ ,  $\omega \equiv 4\pi r^2 a^2 \rho$ , we can write the equations of the system in an autonomous form, which makes the scale invariance of the system transparent:

$$\begin{aligned} \frac{A_{,x}}{A} &= 1 - A + \frac{2\omega}{1 - V^2} \left(1 + \frac{V^2}{3}\right), & \frac{N_{,x}}{N} &= -2 + A - \frac{2\omega}{3}, \\ \frac{\omega_{,s} + (1 + NV)\omega_{,x}}{\omega} + \frac{4\{VV_{,s} + (N + V)V_{,x}\}}{3(1 - V^2)} - \frac{NVA_{,x}}{3A} + \frac{4VN_{,x}}{3} + 2NV \left(1 + \frac{4\omega}{9(1 - V^2)}\right) &= 0, \\ \frac{4V\omega_{,s} + (4V + N + 3NV^2)\omega_{,x}}{\omega} + \frac{4\{(1 + V^2)V_{,s} + (1 + V^2 + 2NV)V_{,x}\}}{1 - V^2} &+ \frac{N(1 - V^2)A_{,x}}{A} + 4(1 + V^2)N_{,x} + 2N(1 + 3V^2) &= 0. \end{aligned} \tag{18}$$

4. *The critical solution.*—One obtains self-similar spacetimes by assuming that  $N$  and  $A$  depend only on  $x$ :  $N = N_{ss}(x)$ ,  $A = A_{ss}(x)$ . Conversely, it can be shown [8] that one can express any spherically symmetric self-similar spacetimes in that form if one retains the freedom of coordinate transformation (17). Then it follows from Eqs. (18) that  $\omega_{ss}$  and  $V_{ss}$  are also functions of  $x$  only:  $\omega = \omega_{ss}(x)$ ,  $V = V_{ss}(x)$ . We fix the coordinate system by requiring that the sonic point (see below) be at  $x = 0$ .

We require (i) that the self-similar solution be analytic for all  $x \in \mathbf{R}$  and (ii) as a boundary condition that the spacetime and the matter are regular,  $A = 1$  and  $V = 0$ , at the center ( $x = -\infty$ ). As has been extensively studied by Ori and Piran [9], the analyticity condition (i) requires that the solution be analytic, in particular, around the sonic point, where the velocity of the fluid particle seen from the observer on the constant  $x$  line is equal to the speed of sound  $1/\sqrt{3}$ . The sonic point is a singular point for the ordinary differential equations (ODE's) satisfied by self-similar solutions, and, considering the power series expansion, one can see that the solutions are specified by one parameter, say, the value of  $V_{ss}(0)$ . This, together with the regularity condition at the center (ii), restricts  $V_{ss}(0)$  to have only discrete values. We employ the Evans-Coleman self-similar solution as our  $H_{ss}$  in the following.

5. *Perturbation.*—Perturbation equations (7) are obtained by taking the first order variation in Eqs. (18) from the Evans-Coleman solution  $H_{ss}$ :

$$h(s, x) = H_{ss}(x) + \epsilon h_{var}(s, x), \tag{19}$$

where  $h$  represents each of  $(N, A, \omega, V)$ . We require  $N_{var}(s, 0) = 0$  to fix the coordinate freedom (17).

We consider eigenmodes of the form  $h_{var}(s, x) = h_p(x)e^{\kappa s}$ , with  $\kappa \in \mathbf{C}$  being a constant. Substituting this form into (19) and then into (18) yields linear, homogeneous first order ODE's for  $(N_p, A_p, \omega_p, V_p)$ .

As in the case of self-similar solutions, we require (i) that the perturbations are analytic for all  $x \in \mathbf{R}$  and (ii) that the perturbed spacetimes are regular at the center ( $A_p, V_p$  vanish at  $x = -\infty$ ). The sonic point becomes a regular singular point for the perturbations. It is not hard to see that apart from the overall multiplicative factor, the perturbation solutions which satisfy the analyticity condition (i) are specified by one free parameter  $\kappa$ . This, together with the regularity condition (ii) at the center, allows only discrete values for  $\kappa$  in general.

Figure 2 shows the profile of the largest relevant eigenmode obtained numerically. It has the eigenvalue  $\kappa \approx 2.81055255$ . This implies the exponent value  $\beta \approx 0.35580192$  from Eq. (15) in our scenario of Sec. 2. This is in good agreement with the value of [3].

To further confirm our scenario, we have checked that there are no other relevant (or marginal) eigenmodes in the range  $0 \leq \text{Re}\kappa \leq 15$ ,  $|\text{Im}\kappa| \leq 14$ . We remark that

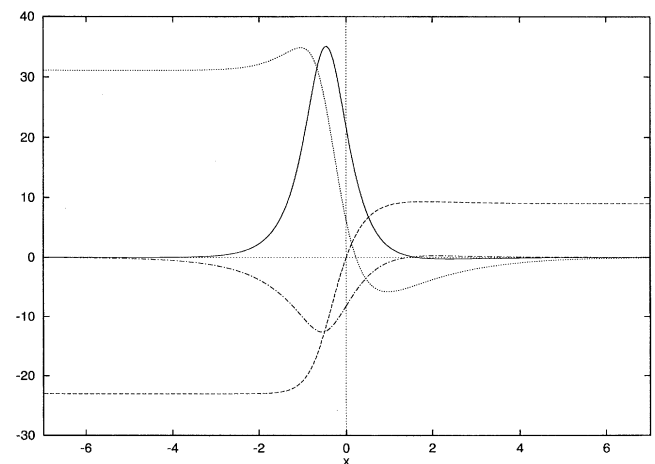


FIG. 2. Profile of the eigenmode with the largest eigenvalue. Curves represent  $A_p$  (solid line),  $N_{ss}^{-1}N_p$  (dashed),  $\omega_{ss}^{-1}\omega_p$  (dotted), and  $V_p$  (dot-dashed).

there is an unphysical “gauge mode” at  $\kappa \approx 0.35699$  (in our gauge), which emerges from a coordinate transformation applied to the self-similar solution. Because of the complicated structure of the equations of motion, we have not found a good argument which can restrict possible eigenvalues (like that of [10], and references therein).

*6. Conclusions and discussions.*—In this Letter we have presented a scenario based on the renormalization group (linear perturbation) ideas, by which the critical behavior in the black hole formation in the radiation fluid collapse is well understood. In particular, we have shown that the critical exponent  $\beta$  is equal to the inverse of the largest Lyapunov exponent  $\text{Re}\kappa$ . We have performed a partial confirmation of the picture, and modulo of some assumptions about distributions of eigenmodes have found an accurate value  $\beta \approx 0.35580192$ , which is also close to that reported in [3].

Complete confirmation of the scenario requires further study of *local* and *global* structures of RG flows around the fixed point  $H_{ss}$ . To establish the local picture, i.e., the contraction property of the RGT on the cospace of our relevant mode in *some* neighborhood of  $H_{ss}$ , one could try to prove that the eigenmodes form a complete set of solutions and that *all* modes except our relevant one are, in fact, irrelevant. Establishing the global picture, which corresponds to proving that the global critical surface exists and the RG flow around it is as depicted in Fig. 1, will pose another challenging problem which will be beyond the scope of linear perturbations. Complete knowledge of the RG flow could expand the horizon of our understanding the gravitational collapse.

It is expected that the critical behavior in the scalar field collapse, where the self-similarity is discrete, can be understood in a manner similar to the analysis in this Letter. One can consider the critical solution as a fixed point of the RGT  $\hat{R}_{s_0}$  with a suitably chosen  $s_0$ , and can perform linear perturbation analysis around it. Work in this direction is now in progress.

Our intuition on gravitational collapse still seems to be heavily based on a few exact solutions, especially the limiting case of pressureless matter. The critical behavior may provide a different limiting case that the final mass is small compared to the initial mass for a more realistic and wider range of matter contents. It will be of great help to settle the problems in gravitational collapse such as cosmic no hair conjecture.

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*Note added.*—To further confirm our scenario we performed a Lyapunov analysis, which extracts eigenvalues  $\{\kappa_1, \kappa_2, \dots\}$  in descending order in  $\text{Re}\kappa_i$ . Our preliminary results strongly support the uniqueness of the relevant mode  $\kappa \approx 2.81$  [8].

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- [6] If there is more than one relevant mode, the stable manifold of  $H_{ss}$  will have a codimension of more than one; a generic family of initial data will not intersect the stable manifold, thus no sharp critical behavior will be seen. Fixed points with more than one relevant mode are responsible for the so-called multicritical behavior.
- [7] An alert reader will notice a gap between the first and second lines of Eq. (10):  $\hat{T}$  on the second line, which is defined as the tangent map *at the fixed point*  $H_{ss}$ , should in fact be the tangent map at  $H_c$ . The maneuver, however, can in general be justified because of Eq. (8). More precisely, what matters most in controlling the critical behavior is the part of the RGT flow in the vicinity of the fixed point (between  $P$  and  $Q$  of Fig. 1), where  $H_{\text{init}}^{(s)}$  spends most of its time during its journey from  $H_{\text{init}}$  to  $H_{\text{init}}^{(s_0)}$ . For the part from  $P$  to  $Q$ , we can justify the use of  $\hat{T}$  above. The parts  $H_{\text{init}} \rightarrow P$  and  $Q \rightarrow H_{\text{init}}^{(s_0)}$  are supposed to be regular. For this reason, it is expected that our conclusion, Eq. (15), is an exact relation.
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