Unpredictability in Some Nonchaotic Dynamical Systems

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We study properties of decay of the correlations for a class of smooth observables and mutual mixing for a class of subsets, with various rates ranging from the power-law to the exponential rate, in simple deterministic nonstrongly chaotic dynamical systems on the torus $\mathbb{T}^2$. In fact, these systems are ergodic, have zero K-S entropy and an absolutely continuous part in their spectrum, and display divergence of trajectories with a power-law rate. We show that they generate time series which are unpredictable in the sense of the statistical theory of prediction.

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In this Letter we consider the relation between dynamics, chaos, and the statistical theory of prediction.

It is now widely admitted that the exponential instability of trajectories implies randomness. In fact, the thermodynamic formalism (see references in Ref. [1]) associates with unstable systems strongly random stationary processes (K systems) which can also be described as Markov processes with the H theorem [2]. This does not mean, however, that dynamical systems which do not exhibit exponential instability may not display some remarkable features of randomness and some features of irreversibility, like entropy production. In this Letter we consider these questions illustrated in a class of transformations on the torus $\mathbb{T}^2$ which display divergence of trajectories with a power-law rate.

The complexity of a dynamical system, generally associated with the instability of the trajectories, can also be statistically described by the properties of some natural invariant measure. Such important properties are the existence of a continuous spectrum and the fast decay of correlations of a class of smooth observables, the ergodicity and the mixing. Another characterization of the complexity is the nonpredictable nature of the time series generated by the dynamical system under observation, at regular time intervals, of orbit visits of the different regions yielding some partition of the phase space. These time series, also called symbolic dynamics, generate a stationary stochastic process. In this respect, the metric and topological entropies are generally considered as measures of randomness of the dynamics. In statistics, the prediction theory [3] gives a characterization of the predictability of some stationary stochastic process when under the observation of the full past of the process; its future can be predicted in the sense of least squares, as briefly summarized in the next section.

The model which we study in this Letter is a simple deterministic dynamical system for which the topological entropy is zero; nevertheless, it has several features of randomness. The main tool which allows this study is the existence of a continuous correlation spectrum of Lebesgue type for a class of observables which we characterize. We display a class of smooth observables for which the rate of decay of correlations is exponential. Although the system is not mixing, we can display a class of subsets which have mixing properties with nonuniform rate, ranging from power-law to exponential decay. We display a class of partitions that generate stationary stochastic processes which are not predictable in the sense of the statistical prediction theory.

Recently, it has been noticed that many nonlinear phenomena in nature emerge at the border separating chaos and order [4]. The properties of the model which is studied here may be pertinent to this type of phenomenon.

A family of maps with zero topological entropy.— The family of invertible transformations $T_{\alpha,p}$ (which we denote simply T, unless in particular cases) acts on the torus $\mathbb{T}^2$, where $\mathbb{T}$ denotes the circle $[0,2\pi]$, $\alpha \in \mathbb{T}$, and p is any nonzero integer. It is defined by

$$T_{\alpha,p}(x) = (x + \alpha, px + y).$$

The normalized Lebesgue measure $d\mu(x,y) = [1/(2\pi)^2]\,dx\,dy$ is invariant under T. It is known [5] that, for irrational α, T has only one invariant measure, so it is ergodic. This family of maps is a special case of the class of the so-called skew product of transformations. Abramov and Rokhlin [6] have given a formula for the computation of the K-S entropy of the skew product. It implies in this case that the K-S and, on account of the unitarity of the invariant measure, also the topological entropy are zero. So, according to a generally admitted terminology, T is not chaotic. Indeed, the dynamics of the system is very simple, as can be seen in the case $\alpha = 0$, where it consists in a rotation with a variable speed x. Nevertheless, we shall see that, although very simple, the system has surprisingly a continuous spectrum which is responsible for some form of loss of memory and displays a divergence of nearby trajectories, which is not of the exponential type. To see this, we compute the iterate T^n, and we obtain $T^n(x,y) = [x + n\alpha, y + npx + n(n-1)p\alpha/2]$. Thus for any

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couple of points (x_1, y_1) and (x_2, y_2) we have

$$T^n(x_1,y_1) - T^n(x_2,y_2) = [x_1 - x_2, y_1 - y_2 + np(x_1 - x_2)] \quad (\mathrm{mod}\ 1);$$

therefore, two arbitrary nearby points having distinct x coordinates will diverge according to a power law.

As to mixing properties of T, it has a delocalizing action on a family of regions of the phase space which we illustrate in Fig. 1 for some simple subsets, in the case $\alpha = 0$ and $p = 1$. Nevertheless, the system is not, in a strict sense, mixing, as we shall see below, that is, $\mu(T^{-n}A \cap B)$ does not converge to $\mu(A)\mu(B)$ as $n \to \infty$, for all subsets A and B of $\mathbb{T}^2$, of positive measure, but only for a class of subsets.

Another quantity which is of great interest is the correlation function of two square integrable observables f and g: $\langle f, U^n g\rangle - \langle f, 1\rangle\langle 1, g\rangle$ where $\langle\,,\rangle$ denotes the standard scalar product in L^2_μ and U is the unitary operator associated with T: $Uf(x,y) = f(T(x,y))$.

We shall first study the decay of correlations and the correlation spectrum for a class of observables. Let us recall the concept of the correlation spectrum of a function f on $\mathbb{T}^2$. With any $f \in L^2_\mu$ is associated a spectral measure $d\mu_f$ which is defined by the relation

$$\langle f, U^n f\rangle = \int_0^{2\pi} e^{in\omega}\, d\mu_f(\omega).$$

The measure $d\mu_f$ is said to be absolutely continuous with respect to the Lebesgue measure if it is continuous with an integrable density $S(\omega)$. S is also called the correlation spectrum of f. If, moreover, $S(\omega)$ is almost everywhere nonzero, $d\mu_f$ is said to be of Lebesgue type.

Now, back to our transformation. We calculate the action of U on the orthonormal basis $\chi_{n,m}(x,y) = e^{inx}e^{iny}$, n and m running in $\mathbb{Z}$, and we obtain from the definition of U $U\chi_{n,m} = e^{in\alpha}\chi_{n+mp,m}$.

Consider first the subspace spanned by the family $\{\chi_{n,m},\ n,m \in \mathbb{Z},\ m \neq 0\}$. This family can be renumbered, so we obtain by multiplying each element by a suitable constant a countable family of orthonormal functions $e_{i,j}$, $i = 1, 2, \ldots$, and $j \in \mathbb{Z}$, such that $Ue_{i,j} = e_{i,j+1}$. It is known [5] that this implies that for any function f in this subspace the spectral measure $d\mu_f$ is absolutely continuous; therefore, we denote this subspace H_{ac}. For every pair of observables in H_{ac} the correlation function decays to zero as $n \to \infty$, according to the Riemann-Lebesgue lemma. On the other hand, the orthocomplement subspace of H_{ac} is spanned by the family of functions $\{\chi_{n,0}\}$, which are all eigenfunctions of U. We call this subspace H_d. It is clear that H_d is the subspace of functions $f \in L^2_\mu$, which depend only on x, so the orthogonal projection operator on H_d is

$$P_d f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,y)\, dy.$$

Thus any observable f such that $P_d f = 0$ belongs to H_{ac}, so that its autocorrelation function decays to 0 as $n \to \infty$. On the other hand, it is possible to express the mixing of two subsets E and F by taking the correlation function of $f = l_E - \mu(E)$ and $g = l_F - \mu(F)$, where l_E is the characteristic function of E:

$$\mu(E \cap T^{-n}F) - \mu(E \cap F) = \langle g, U^n f\rangle.$$

Thus two subsets are mutually mixing if the above quantity tends to zero as $n \to +\infty$. It results, therefore, that all the subsets E such that the function $[l_E - \mu(E)]$ is in H_{ac} are mutually mixing. This is so if this function has zero orthogonal projection on H_d. It comes from the definition of this projection that the section of E at fixed x, $E_x = \{y\colon (x,y) \in E\}$, has a constant length equal to $2\pi\mu(E)$, and this property characterizes the class of mutually mixing subsets.

We now study the problem of the rate of decay of the correlation function. The exponential rate of decay for smooth observables is the signature of a loss of memory. By exponential rate we mean that the correlation function of two observables f and g, with zero expectation, satisfies $|\langle f, U^n g\rangle| \langle Ce^{-\alpha n}\rangle$, $n > 0$, for some positive constants α and C. It is possible to show [7] (we omit the details) that for a *generic class* of continuous functions in H_{ac} the rate of decay is slower than any exponential rate.

There is, nevertheless, a dense class of functions in H_{ac} for which the rate is exponential. An observable $f(x,y)$ in this class is described as follows: (i) f is periodic of period 2π and continuous as a function of (x,y) and (ii) for any fixed value of y, it is, as a function of x, of the form $f(x,y) = F(e^{ix}, y)$, such that the function $z \to F(z,y)$ has analytic continuation in some annulus containing the unit circle $S^1 = \{e^{i\theta},\ \theta \in [0, 2\pi]\}$. The projection of any such function on H_{ac} has an exponentially decaying autocorrelation function; the rate of the exponential decay is related to the width of this annulus.

In general, for less smooth functions, the rate of decay of the correlations follows a power law, which, in some cases, is as slow as $1/n$. This is illustrated by the family of observables which are of the form $f(x,y) = \sum \alpha_i \chi_{E_i}$, where each subset E_i is a union of squares of length $2\pi/m$, with constant total width along y (see Fig. 2). It can be shown (we omit the details) [6] that for these functions the correlation function is either identically vanishing after one iteration of T or decays as slowly as $1/n$. Many examples of such slow decay can be exhibited in this class of observables.

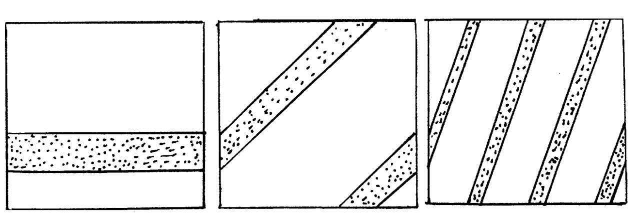

FIG. 1. The action of $T_{0,p}$ on a family of subsets. The figure shows, successively, subsets E, $T_{0,p}E$, and $T^3_{0,p}E$.

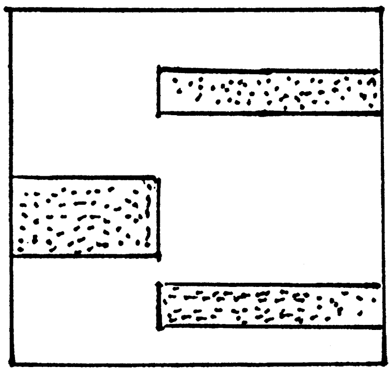

FIG. 2. Subsets with power-law rate of mutual mixing.

Similar properties apply also to mutually mixing subsets. The subsets E_i, described above, have mixing properties with the same rate as the rate of decay of the correlation function.

It is an interesting problem to know if one can find in this system a class of subsets with an exponential rate of mixing.

We may, in fact, describe such a class of subsets which we denote $\mathcal{A}_0$. We introduce a preliminary class of subsets which are constructed as follows. The class of subsets obtained in translating along x an arbitrarily given subset of the y axis Δ_0 by some function $h(x)$, that is, its section Δ_x, is equal to $\Delta_0 + h(x)$ (see Fig. 3). It can be shown [7] that for any pair of translating subsets by the same function $h(x)$ the mixing is achieved after only one iteration; that is, two such subsets E and F satisfy $\mu(E \cap T^{-n}F) = \mu(E)\mu(F)$ for any $n \geq 1$. Therefore, a finite partition into two sets, E and its complement E^c satisfying this equation, is called pairwise independent partition (not to be confused with independent partition).

Now, we call the Δ subset a translation of some Δ_0 by $h(x)$, where $h(x)$ is of the form $H(e^{ix})$ and H has analytic continuation in some annulus containing the unit circle [e.g., $h(x) = a\sin(x)$]. The subsets of the class $\mathcal{A}_0$ are finite unions of disjoint Δ subsets (Fig. 4). It can be shown [7] that the rate of mixing of these subsets is exponential. This implies that for any observable of the form $f(x,y) = \sum \alpha_i \chi_{E_i}$, where $\{E_i\}$ is a partition of the phase space with $E_i \in \mathcal{A}_0$, the correlation function decays exponentially.

Thus, the family of random variables (with respect to the measure $d\mu$) $f_n(x,y) = f(T^n(x,y))$ defines a stationary stochastic process on finite state space with values $\{\alpha_i\}$, which is a factor of the dynamical system (T,μ), so it also has zero entropy. Therefore, this example illustrates a rather strange situation of a finite-state stationary stochastic process with zero entropy and exponentially decaying correlation function. It will be further considered in the next section.

From these examples, we observe that the rate of mutual mixing is related to the smoothness of the border of the subsets. Thus it is expected that a power-law mixing rate could be related to some discontinuity, or even fractality, in the border.

Linear least squares prediction.—In the previous section, we displayed a class of stationary processes f_n taking a finite number of values, with an exponentially decaying correlation function. It is easy to show that if the correlation function of some observable f decays exponentially then the spectral measure $d\mu_f$ is of the Lebesgue type with a density $S(\omega)$ analytic in some annulus containing the unit circle; that is, it is of the form $S(\omega) = G(e^{i\omega})$, where G has analytic continuation in some annulus containing the unit circle. This property has strong consequences in relation to the prediction theory.

In the linear least squares prediction theory [3], one considers a stationary stochastic process $\{f_n\}$, observed in the past, $n, n-1, n-2, \ldots$, and one defines the predictor of the next issue f_{n+1} as the orthogonal projection of f_{n+1} on the closed subspace generated by $f_n, f_{n-1}, f_{n-2}, \ldots$ with the standard scalar product in L^2_μ.

If we denote by P_n the associated orthogonal projection operator, then the predictor of f_{n+1} is $P_n f_{n+1}$. The process is predictable if f_{n+1} is equal with probability 1 to its predictor, that is, if with respect to the norm in L^2_μ $\|f_{n+1} - P_n f_{n+1}\| = 0$. Szëgo gave the expression of the above distance in terms of the correlation spectral measure (see [8]). Let us suppose that the spectral measure of the

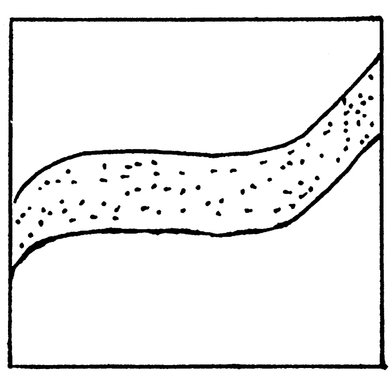

FIG. 3. Translation of a subset Δ_0 by a function $h(x)$.

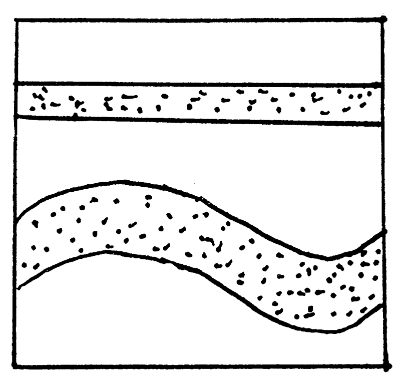

FIG. 4. Subsets with exponential rate of mixing.

process is absolutely continuous with a density $S(\omega)$; then the Szëgo formula gives

$$\|f_{n+1} - P_n f_{n+1}\| = \exp\left(\int_0^{2\pi} \log S(\omega)\, d\omega\right).$$

The left side is equal to zero if and only if the integral of $\log S(\omega)$ over the unit circle is divergent.

In the examples presented in the preceding section, the correlations decay exponentially so that the power spectrum $S(\omega)$ is of the form $S(\omega) = G(e^{i\omega})$, where G is analytic in some annulus containing the unit circle. Therefore, G has a finite number of isolated zeros, $e^{i\omega_1}, e^{i\omega_2}, \ldots, e^{i\omega_s}$. This implies that $S(\omega)$ is continuous and the ω_i's are the only isolated zeros of S. Thus the integral of $\log S(\omega)$ is convergent if it is convergent in the neighborhood of each zero. This is indeed the case, for in this neighborhood we have $G(e^{i\omega}) = (e^{i\omega} - e^{i\omega_i})^{m_i} H_i(e^{i\omega})$, where H_i is continuous and nonvanishing in the neighborhood and m_i is of the order of zero. Thus we may also write $S(\omega) = (\omega - \omega_i)^{m_i} S_i(\omega)$, where S_i is also continuous and nonvanishing in the neighborhood of zero. But $\int \log|\omega - \omega_i|\, d\omega$ is finite around ω_i so the integral of $\log S(\omega)$ is finite around $\omega = \omega_i$. This shows that the processes with exponentially decaying correlations are nonpredictable.

Concluding remarks.—Dynamical systems with positive Kolmogorov-Sinai entropy have an absolutely continuous part in their spectrum of the Lebesgue type with countable multiplicity. Sinai has shown [5] that they, moreover, generate symbolic dynamics (with finite alphabet or finite partition) which correspond to Bernoulli processes. For the transformation $T_{\alpha,p}$ with zero entropy, the system generates stationary processes with finite partitions, not as chaotic as a Bernoulli system, but with decaying correlations and various rates of decay.

Another important problem is to study the evolution of the probability densities which are given here by a Liouville-like equation, $\rho_t((x,y)) = \rho_0(T^{-t}(x,y))$.

One of the outstanding problems in this respect is to define entropy production in terms of the microscopic dynamics. Such an entropy can be associated with a coarse graining which defines a projection operator P of averaging over the cells of some partition. In this respect the main problem is to ensure that the coarse-grained entropy

$$S_t(\rho_0) = -\int P\rho_t((x,y)) \log P\rho_t((x,y))\, dx\, dy$$

is monotonically increasing to an extremum only reached by the uniform density (the so-called Boltzmann H theorem). Such a coarse graining exists naturally in K systems as shown by Goldstein, Misra, and Courbage (see Ref. [9] and references therein) where the cells of the partition are uncountable, each having zero measure.

In Ref. [10], the coarse graining was considered with respect to cells each having positive measure. It turns out that this is the case if the symbolic dynamics generates a stationary process satisfying the Chapman-Kolmogorov equation with an aperiodic transition matrix. It is well known that this equation is satisfied by Markov processes. But a Markov process with aperiodic transition matrix implies positive K-S entropy of the dynamical system. Nevertheless, the Markov property is only sufficient but not necessary for the Chapman-Kolmogorov property, and, furthermore, the stationary stochastic processes which satisfy the Chapman-Kolmogorov equation are non-Markovian, having infinite memory [11]. The partitions which are pairwise independent above noticed in the transformations $T_{\alpha,p}$ are examples of such symbolic dynamics. The coarse graining with respect to this partition leads to an H theorem, which reaches equilibrium after one iteration. It would be interesting to construct Chapman-Kolmogorov partitions which are not pairwise independent for the transformations $T_{\alpha,p}$, or for a more general class of transformations. In general, it can easily be shown that the Chapman-Kolmogorov property with aperiodic transition matrix implies exponential decay of the correlations [12], and, as a consequence, nonpredictability. Therefore, while chaotic dynamics with exponential instability is sufficient to imply an H theorem, the weaker condition of the Chapman-Kolmogorov property enlarges the H theorem to a class of nonchaotic systems. But, as it implies nonpredictability, we have again as a limitation imposed by the irreversibility some intrinsic randomness of the dynamics.

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