PHYSICAL REVIEW LETTERS

VOLUME 74 | 26 JUNE 1995 | NUMBER 26

Waves and Solitons in the Continuum Limit of the Calogero-Sutherland Model

Alexios P. Polychronakos*
Theory Division, CERN, CH-1211, Geneva 23, Switzerland
(Received 9 November 1994)

We examine a collection of particles interacting with inverse-square two-body potentials in the thermodynamic limit. We find explicit large-amplitude density waves and soliton solutions for the motion of the system. Waves can be constructed as coherent states of either solitons or phonons. Therefore, either solitons or phonons can be considered as the fundamental excitations. The generic wave is shown to correspond to a two-band state in the quantum description of the system, while the limiting cases of solitons and phonons correspond to particle and hole excitations.

PACS numbers: 03.40.Kf

There has been much recent interest in the Calogero-Moser-Sutherland model of interacting particles in one dimension [1–3] (which is often referred to in the physics literature as the CS model). This model is related to quantum spin chains with long-range interactions between the spins [4], wave propagation in stratified fluids [5], random matrix theory [2,6], and fractional statistics [7].

The CS model is exactly solvable in both the classical and the quantum regime. Remarkably, the quantum solution is much easier to interpret, exhibiting a straightforward analogy to the free fermion case. In a recent paper, Sutherland and Campbell examined the classical system in the thermodynamic limit and identified the excitations [8]. It was found that the classical system has solitons, corresponding to a single particle running through the rest of them, as well as small-amplitude waves (phonons), identified with holes. The purpose of this paper is to derive large-amplitude wave and soliton solutions of the classical system in the continuous limit, where the particles form a "fluid" and examine their correspondence to the quantum states.

We consider a collection of particles with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} \dot{x}_i^2 + \sum_{i>j} \frac{g}{(x_i - x_j)^2}, \qquad (1)$$

where for convenience we chose them of unit mass. In principle, such a system can be put in a box of length L (with an appropriate modification of the potential [2]). We shall be interested in the limit $N, L \to \infty$ with N/L fixed. In this limit, the system can be described in terms of a density field $\rho(x)$ and a velocity field $v(x)$. At equilibrium, the particles will form a regular lattice of spacing a and density $\rho_0 = 1/a$. The particle current is $J = \rho v$ and by particle conservation

$$\dot{\rho} + \partial J = \dot{\rho} + \partial(\rho v) = 0, \qquad (2)$$

where $\partial = \partial/\partial x$. The kinetic energy of the system is

$$K = \int dx\, \tfrac{1}{2}\rho v^2.$$

We can formally solve Eq. (2) for v to obtain $v = -\partial^{-1}\dot{\rho}/\rho$, and the expression for the kinetic energy becomes

$$K = \int dx\, \frac{(\partial^{-1}\dot{\rho})^2}{2\rho}. \qquad (3)$$

This is exactly the kinetic term of the collective field Hamiltonian description of a many-body system [9]. The potential energy can also be expressed in terms of the density. The naive expression, however, which would be

$$V = \int dx\, dy \frac{g}{2}\, \frac{\rho(x)\rho(y)}{(x-y)^2}$$

is incorrect. The reason is that the interaction is singular at coincidence points, and thus a substantial part of the potential energy comes from nearest neighbors and is not accurately reproduced by the naive continuous expression. The correct expression requires a careful conversion of the discrete sum in terms of the continuous fields.

0031-9007/95/74(26)/5153(5)$06.00

Alternatively, we can simply take the classical limit ($\hbar \to 0$) of the quantum mechanical expression derived in the collective field formulation [10]. The result is

$$V = \int dx \left\{ \frac{\pi^2 g}{6} \rho^3 - \frac{g}{2} \rho \partial \tilde{\rho} + \frac{g}{8} \frac{(\partial \rho)^2}{\rho} \right\}, \quad (4)$$

where $\tilde{\rho}$ stands for the Hilbert transform

$$\begin{aligned} \tilde{\rho} &= \int dy \, \mathrm{P.P.} \frac{1}{x-y} \rho(y) \\ &= \tfrac{1}{2} \lim_{\epsilon \to 0} \int dy \left(\frac{1}{x - y + i\epsilon} + \frac{1}{x - y - i\epsilon} \right) \rho(y). \end{aligned} \quad (5)$$

The first term, which accounts for the interaction of each particle with its few nearest neighbors, is the dominant one in the limit where the scale of variation of ρ is much larger than the lattice spacing. In our case, however, we are interested in finite-width fluctuations, and we must keep the full expression.

The dynamics of the system can be found by varying the Lagrangian $L = K - V + \mu\rho$ with respect to ρ. The chemical potential μ plays the role of a Lagrange multiplier ensuring that the total number of particles remains constant. The resulting equations of motion are

$$-\partial^{-1}\dot{v} - \tfrac{1}{2} v^2 - \frac{\pi^2 g}{2} \rho^2 + g \, \partial \tilde{\rho} + \frac{g}{8} \left(\frac{\partial \rho}{\rho} \right)^2 + \frac{g}{4} \partial \left(\frac{\partial \rho}{\rho} \right) + \mu = 0, \quad (6)$$

as well as Eq. (2). The inverse derivative operator in Eq. (6) is defined in terms of the principal value in Fourier space $\partial^{-1} = \lim_{\epsilon \to 0} k/(k^2 + \epsilon^2)$. In particular, acting on a constant it gives zero. By requiring that the static configuration $v = 0$, $\rho = \rho_0$ be a solution of Eq. (6), we obtain the value of the chemical potential

$$\mu = (\pi^2 g/2) \rho_0^2. \quad (7)$$

This is in agreement with the value obtained from the exact solution of the many-body problem [2,8].

Small-amplitude waves.—From the above equations we can obtain the dispersion relation in the linearized regime of small-amplitude waves, which we shall call phonons. Noting that the Fourier transform of $\partial \tilde{\rho}$ is $\pi |k| \rho(k)$, we obtain

$$\mathrm{v}_{\mathrm{phase}}^2 = (\omega/k)^2 = (g \pi \rho_0 - |k|/2)^2$$

or

$$\omega = \sqrt{g} (\pi \rho_0 |k| - k^2/2). \quad (8)$$

From Eq. (8) we deduce that the velocity of sound v_s, defined as the phase (or group) velocity in the long-wavelength limit, is

$$\mathrm{v}_s = \pi \rho_0 \sqrt{g}. \quad (9)$$

In terms of the group velocity v_g the dispersion relation becomes

$$\omega = (\mathrm{v}_s^2 - \mathrm{v}_g^2)/2\sqrt{g}. \quad (10)$$

We observe that Eqs. (9) and (10) are the exact results. The group velocity is always smaller than the velocity of sound, and the above linearized waves can be identified with holes in the quantum theory. Notice that the above formulas are valid for $|k| < \pi\rho_0 = \pi/a$, else the group velocity turns negative. This is reasonable, since the above condition restricts the momentum to the fundamental region of the Brillouin zone, thus avoiding umklapp.

Solitons.—As observed in Ref. [8], the many-body system should exhibit soliton solutions, corresponding to particle excitations. On the other hand, in Ref. [11] an equation similar to Eq. (6) was written for a system of free fermions, coming from an effective Lagrangian. This equation has solitary wave solutions [11]. As we will demonstrate here, our Eqs. (6) and (2) also have solitary wave solutions of a rational type; we shall call these solutions solitons and will comment later on their true nature. For a localized constant profile configuration, propagating at speed v, both ρ and v are functions of $x - \mathrm{v}t$ only. From Eq. (2) we have

$$\partial(v\rho - \mathrm{v}\rho) = 0, \qquad v = \frac{\rho - \rho_0}{\rho} \mathrm{v}. \quad (11)$$

In the above, the integration constant is fixed by the boundary condition that $v \to 0$ at $x \to \pm\infty$, where $\rho \to \rho_0$. Similarly, Eq. (6) becomes

$$\frac{\mathrm{v}^2}{2} \left(\frac{\rho_0^2}{\rho^2} - 1 \right) + \frac{\pi^2 g}{2} (\rho^2 - \rho_0^2) - g \, \partial \tilde{\rho} - \frac{g}{8} \left(\frac{\partial \rho}{\rho} \right)^2 - \frac{g}{4} \partial \left(\frac{\partial \rho}{\rho} \right) = 0. \quad (12)$$

To guess a solution for Eq. (12) of the form $\rho_{\mathrm{sol}} = \rho_0 + \delta\rho$, where $\delta\rho$ is localized, we notice that the term in Eq. (12) containing the Hilbert transform will always produce out of a localized function a tail falling off quadratically. Thus, $\delta\rho$ itself should have such a behavior at infinity. The simplest function of this form is

$$\rho_{\mathrm{sol}} = \rho_0 + A/(x^2 + B^2).$$

Plugging the above form into Eq. (12) we find, after an amount of algebra, that it is indeed a solution, provided that $\mathrm{v} > \mathrm{v}_s$ and

$$A = \frac{u}{\pi^2 \rho_0}, \qquad B = \frac{u}{\pi \rho_0}, \qquad u = \frac{\mathrm{v}_s^2}{\mathrm{v}^2 - \mathrm{v}_s^2}. \quad (13)$$

We finally arrive at the soliton profile

$$\rho_{\mathrm{sol}} = \rho_0 \left(1 + \frac{u}{(\pi \rho_0 x)^2 + u^2} \right), \qquad u = \frac{\mathrm{v}_s^2}{\mathrm{v}^2 - \mathrm{v}_s^2}. \quad (14)$$

The above solution is, strictly speaking, a solitary wave. Since the initial many-body system (1) is integrable, we expect the corresponding continuum system to be also integrable, although a direct proof is lacking, and thus Eq. (14) to be a true soliton. This is corroborated by the correspondence of these solutions to particles, as demonstrated below.

The above soliton carries particle number Q, momentum P, and energy E, defined as the extra amount over the static solution ρ_0. We find

$$\begin{aligned} Q &= \int dx\,(\rho_{\rm sol} - \rho_0) = 1\,, \\ P &= \int dx\,\rho_{\rm sol} v = {\rm v}\,, \\ E &= \int dx[K(\rho_{\rm sol}) + V(\rho_{\rm sol}) - V(\rho_0)] = \tfrac{1}{2}{\rm v}^2. \end{aligned} \tag{15}$$

We observe that the net particle number carried by soliton is 1, independent of its velocity; its momentum and energy are also those of a free particle of unit mass moving at the soliton velocity v. Therefore, the soliton can be exactly identified with a particle excitation of the system. Again, this is in agreement with exact results drawn from the quantum theory, where particle excitations always move faster than sound [8]. Notice, further, that the solitons become thinner as their velocity increases, while their spread diverges as they slow down to the velocity of sound.

The above result for Q implies that the displacement of the equilibrium lattice far away from the soliton is $\pm$ half lattice spacing either way (so that there is an excess of one particle near the soliton). This result as well as the form of the soliton (14) is at odds with the results found in Ref. [8]. We suspect that the source of the discrepancy is the truncation to a finite number of x derivatives of the form for the potential in Ref. [8]; this turns the equation to a local one and gives the soliton an exponential decay, rather than the inverse-square decay of the nonlocal equation. We also notice that our soliton has some important qualitative differences from the solitons in the semiclassical fermion theory of Ref. [11]: Our solitons carry a positive particle number of 1, as opposed to a negative particle number in Ref. [11], which would rather identify them as holes. Further, there are no static solitons in our case, since $|{\rm v}| > {\rm v}_s$, while in Ref. [11] solitons can slow down to zero speed. Finally, the definition of momentum used in Ref. [11] differs from ours by a surface term. Clearly Eq. (15) is the physically sensible definition in our case.

Finite-amplitude waves.—Soliton profiles moving at very large distances from each other will obviously remain solutions. If we could form a state consisting of a sequence of solitons at regular distances spaced by λ, all moving with the same velocity v, we would have found a large-amplitude wave solution with wavelength λ. We thus try the form

$$\begin{aligned} \rho_{\rm wave} - \rho_0 &= \sum_{n=-\infty}^{\infty} [\rho_{\rm sol}(x - n\lambda) - \rho_0] \\ &= \frac{1}{\lambda}\frac{\sinh(2u/\lambda\rho_0)}{\cosh(2u/\lambda\rho_0) - \cos(2\pi x/\lambda)}\,, \end{aligned} \tag{16}$$

where now the parameter u is *not* necessarily given by ${\rm v}_s^2/({\rm v}^2 - {\rm v}_s^2)$, since the proximity of the other solitons may have changed their common velocity. The above wave form is characterized by its amplitude A, defined as midway the distance from peak to trough,

$$A = \frac{\rho_{\max} - \rho_{\min}}{2} = \frac{1}{\lambda\sinh(2u/\lambda\rho_0)}\,, \tag{17}$$

as well as by its wavelength λ. Substituting the form (16) into Eq. (12) we find, again after quite a bit of algebra, that it is indeed a solution provided

$$\tanh\frac{2u}{\lambda\rho_0} = \frac{2\lambda\rho_0{\rm v}_s^2}{\lambda^2\rho_0^2({\rm v}^2 - {\rm v}_s^2) - {\rm v}_s^2}\,. \tag{18}$$

The above is the amplitude-dependent dispersion relation for the nonlinear waves of the system. Before we interpret it, however, we must note the following: The conventions used for deriving Eq. (12) were that the solution ρ carries some particle number and momentum on top of the "vacuum" solution ρ_0. This is reasonable for an isolated soliton, but rather inconvenient for a wave solution, which is thought to be a fluctuation carrying no net particle number and no net momentum (no drift). But the presence of the solitons in Eq. (16) adds one particle per length λ, and thus the true equilibrium density of the system is $\rho_0 + 1/\lambda$. Further, the solitons contribute a momentum v per length λ; to neutralize it, we must boost the whole system in the opposite direction by an appropriate amount. After performing these redefinitions, the expression for the wave in terms of the true velocity v and true background density ρ_0 is

$$\rho_{\rm wave} = \rho_0 + \frac{1}{\lambda}\left(\frac{1}{\sqrt{\lambda^2A^2 + 1} - \lambda A\cos(2\pi x/\lambda)} - 1\right) \tag{19}$$

and the nonlinear dispersion relation in terms of the amplitude A is

$${\rm v} = \frac{\omega}{k} = \left({\rm v}_s - \frac{\pi\sqrt{g}}{\lambda}\right)\sqrt{1 + \frac{2A^2(\lambda\rho_0 - 1)}{\rho_0^2(1 + \sqrt{\lambda^2A^2 + 1})}}\,. \tag{20}$$

In the limit $\lambda \to \infty$ the above equations reduce to the single soliton solution. In the limit $A \to 0$, on the other hand, the above formulas become

$$\begin{aligned} \rho_{\rm wave} &= \rho_0 + A\cos kx\,, \qquad k = 2\pi/\lambda\,, \\ {\rm v} &= \omega/k = {\rm v}_s - (\sqrt{g}/2)\,k\,, \end{aligned} \tag{21}$$

which is the small-amplitude wave solution and dispersion relation. We see, therefore, that the above solutions interpolate between the two extreme cases. We stress that the generic wave can run either faster or slower than the speed of sound.

In summary, we have found exact soliton and wave solutions for the CS system in the continuum limit. Certainly the above do not exhaust the list of solutions; the general motion of the system will be a nonlinear superposition of waves (or solitons).

It is instructive to put the above solutions into correspondence with the quantum mechanical states. Consider N particles in a space of length L. The ground state of the system consists of a "Luttinger sea" in the pseudomomentum, with spacing between adjacent particles equal to $2\pi\ell/L$ and "Fermi level" $\pi\ell N/L$, where $g = \ell(\ell - \hbar)$. At the limit $\hbar \rightarrow 0$, $N, L \rightarrow \infty$, $N/L \rightarrow \rho_0$, the ground state becomes a continuous filled band with Fermi level $P_F = \pi\sqrt{g}\,\rho_0$. A small-amplitude wave, corresponding to a hole, is a very small gap in the band. A soliton, corresponding to a particle excitation, is a single particle peeled from the Fermi level and placed some distance above. The generic finite-amplitude wave corresponds to a state with *two* continuous filled bands of widths P_1 and P_2 (with $P_1 + P_2 = 2P_F$) and with a gap G between them. These are related to the wave parameters as

$$\lambda = \frac{2\pi\sqrt{g}}{P_1}, \qquad \mathrm{v} = \frac{P_2}{2}\left(\frac{G}{\pi\sqrt{g}\,\rho_0} + 1\right). \tag{22}$$

Such a state can be visualized as arising either by successively exciting single particles by the same constant momentum until they form a continuous band or by gradually augmenting the gap of a hole until it becomes finite. This state can thus be thought of either as a coherent state of solitons (much like the way we constructed the wave solution) or as a coherent state of phonons, their nonlinear nature accounting for the change in profile as they accumulate. Indeed, the soliton itself can be thought of as a superposition of many phonons with very large wave number and the phonon as a superposition of many solitons just above the Fermi level. For the finite N (finite L) system the distinction between the two is fuzzy and, in principle, only one kind of excitations need be considered as fundamental. Note, further, that quantum mechanically the holes behave as particles with fractional statistics of order $\hbar/\ell$ (meaning that $\ell/\hbar$ of them put together would form a fermion). At the classical limit $\hbar \rightarrow 0$, thus, they become bosons, as they should be since phonons obey no exclusion principle. Particles, on the other hand, carry statistics of order $\ell/\hbar$. Thus in the classical limit they become "superfermions," meaning that no two of them can occupy relatively nearby quantum states. This is consistent with the inverse-square repulsion between the classical particles.

The above results are of direct relevance to the large-N limit of one-dimensional free matrix models. The particular wave and soliton solutions correspond to motions of the density of eigenvalues in the unitary and Hermitian models, respectively. Taking, for clarity, the Hermitian case, the motion of a free $N \times N$ matrix M with angular momentum ℓ is

$$M_{jk} = \delta_{jk}(p_j t + a_j) + (1 - \delta_{jk})\frac{i\ell}{p_j - p_k}. \tag{23}$$

The situation where most of the eigenvalues lie on a regular lattice with only one of them moving with velocity v is reproduced by choosing

$$p_j = \frac{2\pi\ell}{a(N-2)}\left(j - \frac{N}{2}\right) \quad (\text{for } j < N), \qquad p_N = \mathrm{v}, \quad a_j = 0. \tag{24}$$

(Notice that the above momenta $p_1, \ldots, p_{N-1}$ span the values between the two Fermi levels $\pm\pi\ell/a$.) It should be possible to prove analytically that the eigenvalues of Eq. (23) with parameters (24) have a density given by our soliton solution, but in practice this is a very hard task. The corresponding problem for unitary matrices is even harder to tackle, while our wave (19) readily provides the solution. Many-soliton solutions will be given by eigenvalues of Eq. (23) with, now, more then one of the momenta p_j taking values equal to the velocities of the solitons, while the rest span the Luttinger sea.

We conclude by noting that the quantum mechanical problem separates into two noninteracting chiral sectors having to do with excitations near either end of the Luttinger sea. (The two sectors mix nonperturbatively when a number of particles of order N is excited, depleting the sea.) Therefore, Eq. (6) governing the continuum system should also decompose into two nonmixing, first order in time equations, one for each sector. For the corresponding equation for free fermions this is indeed the case [12]. In fact, from the collective field description of the system when only one chiral sector is present [13], we deduce that the chiral equations are exactly of the Benjamin-Ono type [5]. The exact field combinations in terms of which this decomposition would be achieved, however, are not known and constitute an open problem.

I would like to thank Bill Sutherland and Joe Minahan for discussions, and the Aspen Center of Physics for its hospitality during summer 1994, where this work was started.

*Present address: Theoretical Physics Dept., Uppsala University, S-751 08 Uppsala, Sweden. Electronic address: poly@calypso.teorfys.uu.se

[1] F. Calogero, J. Math. Phys. (N.Y.) **10**, 2191 and 2197 (1969); **12**, 419 (1971).

[2] B. Sutherland, Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972); Phys. Rev. Lett. **34**, 1083 (1975).
[3] J. Moser, *Dynamical Systems, Theory and Applications,* Lecture Notes in Physics Vol. 38 (Springer-Verlag, New York, 1975).
[4] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988); B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
[5] H. H. Chen, Y. C. Lee, and N. R. Pereira, Phys. Fluids **22**, 187 (1979).
[6] B. D. Simons, P. A. Lee, and B. L. Altshuler, Phys. Rev. Lett. **70**, 4122 (1993); **72**, 64 (1994); E. Mucciolo, B. Shastry, B. Simons, and B. Altshuler, Report No. cond-mat/9309030 (to be published).
[7] J. M. Leinaas and J. Myrheim, Phys. Rev. B **37**, 9286 (1988); A. Polychronakos, Nucl. Phys. **B234**, 597 (1989); F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991); Y.-S. Wu, Phys. Rev. Lett. **73**, 922 (1994), S. B. Isakov, Int. J. Mod. Phys. **A9**, 2563 (1994).
[8] B. Sutherland and J. Campbell, Phys. Rev. B **50**, 888 (1994).
[9] N. N. Bogoliubov and D. N. Zubarev, JETP Lett. **1**, 83 (1955); A. Jevicki and B. Sakita, Nucl. Phys. **B165**, 511 (1980).
[10] I. Andric, A. Jevicki, and H. Levine, Nucl. Phys. **B215**, 307 (1983); I. Andric and V. Bardek, J. Phys. A **21**, 2847 (1988).
[11] A. Jevicki, Nucl. Phys. **B376**, 75 (1992).
[12] J. Polchinski, Nucl. Phys. **B362**, 125 (1981).
[13] J. Minahan and A. P. Polychronakos, Phys. Rev. B **50**, 4236 (1994).