No Perfect Two-State Cellular Automata for Density Classification Exists

Mark Land and Richard K. Belew
Department of Computer Science and Engineering, University of California, San Diego, La Jolla, California 92093-0114
(Received 9 January 1995)

Recently there have been many attempts to evolve one-dimensional two-state cellular automata which classify binary strings according to their densities of 1's and 0's. The current best-known approaches involve particle-based systems of information transfer. A proof is given that there does not exist a two-state cellular automata which performs the task perfectly. This is true even in multiple dimensions.

PACS numbers: 89.80.+h, 02.70.Rw, 47.11.+j

Since their inception, cellular automata (CA) have proven to be an extremely austere yet powerful class of algorithmic specifications. A number of investigations into the relationship between computational systems and the physical world have used CA as their foundation [1–3], so it is reasonable that recent extensions of this work to the role of evolution in the physical and computational worlds would also rely upon CA.

Beyond simple understanding the limits of CA computation, a central motivation for this work concerns the potential of CA as models of biological phenomena such as ontogenesis. A defining property of CA is the way in which their computations depend upon and affect an arbitrarily large *global* state field, despite the fact that all of their operations are defined over a finite, *local* neighborhood. This can be related to the development of a multicellular organism involving ensembles of billions of cell types; many of the interactions between these cells are "local," but each cell develops from and functions according to a single, shared, and hence "global" genome. Moreover, the ensemble is evaluated in terms of the "fitness" of the single organism. In this context, the use of CA as a model of such "emergent computations" is especially appropriate [4].

A canonical example of such an emergent computation is to use a locally specified CA to determine the global density of bits in an initial state configuration. More specifically, given an arbitrary initial configuration of a one-dimensional two-state CA, the CA should converge to a state of all 1's if the initial configuration contains a density of 1's $\geq \rho$, and to all 0's otherwise, for some ρ between 0 and 1. In what follows, we will assume for convenience that if n is the size of the one-dimensional lattice, ρn is not an integer.

The best known solution to this problem for $\rho = 1/2$ is the Gaks-Kurdyumov-Levin (GKL) rule [5]. The basic strategy taken by GKL can be viewed as consolidating regions of local homogeneity, and sending "signals" to ambiguous regions. Solid blocks of 0's or 1's convert adjacent undecided areas to their own state. Signals are sent in the form of stable "particle" structures that propogate along the string. In this way local information can be transmitted to far away parts of the CA. The GKL rule is almost always successful on configurations with high or low densities, but there is a combinatorial explosion of configurations with densities very close to 1/2, and these give GKL considerable trouble.

Attempts to evolve CA that solve this task have occupied a number of investigators. Mitchell, Hraber, and Crutchfield [6] used the genetic algorithm to find solutions to this problem for $\rho = 1/2$. They were successful in finding solutions that were nearly as good as GKL. Solutions were tested by picking random initial configurations with a uniform distribution over densities. This eliminated the problem of the combinatorial explosion of cases with density close to 1/2. With this distribution, their best solution was correct about 95% of the time, compared to 97.8% for GKL, on lattices of size 149 (more recent work by Mitchell has achieved better results, but still no better than GKL [7]).

Experiments of our own evolving CA for the density classification task using a different representation have resulted in an automaton that may be in some ways superior to, but is roughly comparable with, that found by Mitchell. Our solution was correct 97.8% of the time on the uniform density distribution test described above, comparable to GKL. A major advantage of our solution is that it converges much faster than GKL on many initial configurations. Our difficulties in evolving even better solutions led us to wonder whether such a CA actually exists. The central result of this paper is that for a one-dimensional lattice of fixed size n, and for a fixed $r \geq 1$, there exists no two-state CA rule defined over radius r which correctly classifies all possible initial configurations.

Theorem.—For a given neighborhood radius r, density ρ, and lattice size n such that $N > \max[4r/\rho, 4r/(1-\rho)]$, there does not exist a two-state CA rule of radius r which correctly classifies every configuration of size n for density ρ.

The proof will be by contradiction. First we will assume a perfect CA rule exists, and we will use lemmas to consider a sequence of perverse initial configurations which cannot all be handled correctly by it. Let R^*_ρ denote some CA rule which is perfect for doing ρ classification on lattices of size n. We begin by considering the behav-

 0031-9007/95/74(25)/5148(3)$06.00

ior of R_ρ^* on "solid" blocks of 1's or 0's, then demand that the perfect rule never causes the lattice density to cross the ρ threshold. We then focus on configurations of the general form 0^i1^j, where, of course, the most important CA transitions involve the discontinuity between 0 and 1 solid blocks, especially when we are near the threshold $j/n = \rho$ density. Finally, we consider the behavior of R_ρ^* on configurations where the critical additional 1 pushing density above ρ is buried within the solid block of 0's.

Lemma 1.—Given any configuration in which the density of 1's is less than (or greater than) ρ, R_ρ^* will keep the density of 1's less than (greater than) ρ at the next cycle, and for all future cycles.

Proof: For density $<\rho$, note that R_ρ^* will eventually take any configuration with a density greater than ρ to all 1's. Therefore, if a configuration with density *less than* ρ is taken to a configuration with density *greater than* ρ, it will eventually be classified as all 1's, which is incorrect. This contradicts the perfectness of R_ρ^*, and so proves the lemma.

The proof for density $>\rho$ proceeds analogously.

Lemma 2.—In a given configuration, if a cell and its r neighbors on each side are all 0 (or all 1) then at the next cycle the cell will still be 0 (or 1).

Proof: Consider the configuration which is all 0's (or all 1's). By the definition of the problem, this configuration must remain unchanged at the next cycle (under the action of R_ρ^*). Since this must also be true of every length $2r + 1$ substring of the configuration, the lemma is proven.

Lemma 3.—Given the configuration

$$\sigma_1 = 0^i1^j \tag{1}$$

with $i, j \geq 2r$ and $i + j = n$, number of cells 0 to $n - 1$ (so that cells 0 to $i - 1$ are 0's and cells i to $n - 1$ are 1's). Then at the next cycle, cells r to $i - r - 1$ will be 0, and cells $i + r$ to $n - r - 1$ will be 1.

Proof: This lemma follows immediately from Lemma 2.

Lemma 4.—There exist binary strings α, β, and γ, with lengths r, $2r$, and r, respectively, such that for any i and j with $i, j \geq 2r$ and $i + j = n$, R_ρ^* will take 0^i1^j to $\alpha 0^{i-2r}\beta 1^{j-2r}\gamma$ on the next cycle.

Proof: Lemma 3 guarantees that cells r to $i - r - 1$ will be 0's and cells $i + r$ to $n - r - 1$ will be 1's. All that is left to show is that α, β, and γ are the same for all satisfying choices of i and j. First consider β, which contains the values of cells $i - r$ to $i + r - 1$. The most extreme elements of β, cells $i - r$ and $i + r - 1$, themselves depend only on neighborhoods also of radius r. Hence the value of β is completely determined by the values of cells $i - 2r$ to $i + 2r - 1$ at the previous cycle. But for any allowed i and j, these cells contain $0^{2r}1^{2r}$. Therefore β is independent of the values of i and j. The same argument applies for α and γ.

Lemma 5.—Given any $\rho \leq 1/2$, $n \geq 4r/\rho$, and i, j such that $i, j \geq 2r$, $i + j = n$, R_ρ^* will take 0^i1^j to $\alpha 0^{i-2r}\beta 1^{j-2r}\gamma$ on the next cycle, where the string $\alpha\beta\gamma$ has exactly $2r$ 0's and $2r$ 1's.

Proof: Consider the special case of configuration σ_1 for which $j = \lfloor \rho n \rfloor$. In other words, there are exactly enough 0's so that the density of 1's is less than ρ. In the next cycle, R_ρ^* must keep the density of 1's below ρ (by Lemma 1), and so the number of 0's must stay the same or increase in the next cycle. As long as i and j are both $\geq 2r$, we can apply Lemma 4. This tells us that 0^i1^j is taken to $\alpha 0^{i-2r}\beta 1^{j-2r}\gamma$ in one cycle. Therefore, it must be the case that the number of 0's in $\alpha\beta\gamma \geq 2r$, in order for the total number of 0's to be $\geq i$. Note that if $\rho n > 2r$, then $i, j \geq 2r$, because $\rho \leq 1/2$.

Now consider the complementary case $0^{i'}1^{j'}$, where $j' = \lfloor \rho n \rfloor + 1$ (and $i' + j' = n$); there is exactly one more 1 and one less 0 than in the above configuration, and now the density is greater than ρ. This time the density must be kept *above* ρ, and so the number of 0's must stay the same or *decrease* in the next cycle. If we again assume $\rho n > 2r$ then $i', j' \geq 2r$, and we can apply Lemma 4. This time we deduce that the number of 0's in $\alpha\beta\gamma \leq 2r$. Taken with the previous paragraph, and noting that $|\alpha\beta\gamma| = 4r$, this completes the proof of the lemma.

Lemma 6.—Applying R_ρ^* to $0^{n-1}1$ for a single cycle results in a string which has either zero 1's or more than one 1.

Proof: Assume that the resulting string had a single 1. Then this would be the same as the original string, perhaps shifted to the left or right. Therefore applying R_ρ^* again would have the same effect, and there would be exactly one 1 after two cycles, and again after three cycles, etc. At any time, there would be a single 1, and so the computation would never converge to all 0's, as required by the problem definition.

Consider the configuration that results from changing a single 0 to a 1 in the middle of the 0 block in σ_1:

$$\sigma_2 = 0^{2r}10^{i-1}1^j. \tag{2}$$

Let $j = \lfloor \rho n \rfloor$ and $i + j = n$. Also assume n is large enough that $i - 1 - 2r \geq 2r$. This will be the case if $(1 - \rho)n > 4r$. Then this string has just enough 1's to make the density greater than ρ. Compare this to σ_1, which has just enough 0's to make the density less than ρ.

σ_2 is exactly the same as σ_1, except for the cell in position $2r$. Therefore, after one cycle, the two strings will be the same, except for cells r to $3r$, which are the only cells that can be affected by position $2r$. But by Lemma 4, σ_1 will be taken to $\alpha 0^{i-2r}\beta 1^{j-2r}\gamma$ in one cycle. This means σ_2 will be taken to

$$\sigma_3 = \alpha\delta 0^{i-1-4r}\beta 1^{j-2r}\gamma, \tag{3}$$

where δ is some string of length $2r + 1$. (We know that $i - 1 - 4r \geq 0$ because we specified that $i - 1 - 2r \geq 2r$.)

We gain insight into what δ is by considering the very simple configuration $0^{n-1}1$. By Lemma 6, there are two possibilities. Either $0^{n-1}1$ is taken immediately to all 0's or it is taken to a string with multiple 1's on the first cycle.

Assume it is immediately taken to all 0's. This means that in any configuration in which there is substring of cells $0^{2r}10^{2r}$, the middle $2r + 1$ cells will become all 0's at the next cycle. This, of course, applies to σ_2, and tells us that $\delta = 0^{2r+1}$ in σ_3. But this implies that σ_2 and σ_1 are both taken to $\alpha 0^{i-2r}\beta 1^{j-2r}\gamma$, which is a contradiction, since they have densities on either side of ρ and must be classified differently. Therefore our assumption that $0^{n-1}1$ is taken immediately to all 0's must be wrong.

The only other possibility is that $0^{n-1}1$ is taken to a string with multiple 1's in a single cycle. These 1's must all be in cells within r of position $n - 1$, due to Lemma 2. And, in fact, in any configuration in which there is the substring $0^{2r}10^{2r}$, the middle $2r + 1$ cells will contain more than one 1 at the next cycle.

Finally, we are in a position to consider a simple but deadly configuration for R^*_ρ. Consider

$$\sigma_4 = 0^{2r}10^{i-2r}1^{j-1} \tag{4}$$

[again with $(1 - \rho)n > 4r$]. This has just enough 0's so that the density is less than ρ. By using Lemma 4 and arguing as we did for σ_2, we see that after one cycle σ_4 will become

$$\sigma_5 = \alpha\delta' 0^{i-4r}\beta 1^{j-1-2r}\gamma, \tag{5}$$

where δ' is a string of length $2r + 1$. We can see from the preceding paragraph that δ' must contain at least two 1's. Using Lemma 5 we can conclude that the total number of 1's in σ_5 is $\geq (j - 1 - 2r) + 2r + 2 = j + 1$. But $j + 1 = \lfloor \rho n \rfloor + 1$, so the density of σ_5 is greater than ρ, which contradicts Lemma 1.

We have shown that no matter what R^*_ρ does to $0^{n-1}1$, it leads to a contradiction and the theorem quickly follows.

Proof (of theorem): For $\rho \leq 1/2$, if we take $\rho n > 4r$ then the assumptions made in the discussion above [specifically, $\rho n > 2r$ and $(1 - \rho)n > 4r$] are valid, and we are done. For $\rho > 1/2$, the proof is analogous, and the condition on n is $(1 - \rho)n > 4r$.

Note that the restriction on the size of n in the theorem is not tight, and, in fact, for $\rho \leq 1/3$, the proof is given requires only that $n > (2r + 1)/\rho$. Likewise for $\rho \geq 2/3$, it is sufficient to have $n > (2r + 1)/(1 - \rho)$.

Although the proof given is for one-dimensional CA, it can easily be extended to the multidimensional case. If the initial state values vary along only one of the dimensions, this property must be maintained at each successive time step. The argument used for the one-dimensional case then applies to the entire n-dimensional lattice.

In conclusion, this paper proves that no two-state CA can do the density classification task perfectly. However, it says nothing about how well an imperfect CA might be able to do. Empirically, the best known solutions seem to correctly classify only about 80% of all possible initial configurations (this is in contrast to the uniform *density* distribution test described earlier), with performance worsening for larger lattices. Short of perfect density classification, does there exist a theoretical bound for two-state CA?

A related question concerns the nature of the mistakes made. Informal experiments suggest that the GKL rule (and the evolved rules) misclassifies only configurations whose densities are within some ϵ of 0.5, with ϵ decreasing as lattice size increases. If this ϵ can be made small enough, then the CA can be used as a sort of measurement device. In general, do there exist CA rules which err only within ϵ of ρ? What are the bounds on how small ϵ can be, as a function of lattice size?

It would also be interesting, though probably much more difficult, to generalize the results of this paper to CA with more than two states. Is there a number k, independent of the width of the lattice, such that some CA with k states can solve the problem for all cases on arbitrarily large lattices? We are especially interested in situations where only some of each cell's state bits are "visible" to neighboring cells, and the rest hidden. For example, with only one visible bit, this extension is very similar to the situation dealt with in this paper, but where each cell is allowed some memory.

[1] C. Langton, Physica (Amsterdam) **42D**, 12–37 (1990).
[2] *Cellular Automata Machines: A New Environment for Modeling,* edited by T. Tommaso and N. Margolus (MIT Press, Cambridge, 1987).
[3] *Theory and Applications of Cellular Automata,* edited by S. Wolfram (World Scientific, Singapore, 1986).
[4] *Emergent Computation: Self-organizing, Collective, and Cooperative Phenomena in Natural and Artificial Computing Networks,* edited by S. Forrest (MIT Press, Cambridge, 1991).
[5] P. Gaks, G.L. Kurdyumov, and L.A. Levin, Probl. Peredachi. Inform., **14**, 92–98 (1978).
[6] M. Mitchell, P.T. Hraber, and J.P. Crutchfield, Complex Systems **7**, 89–130 (1993).
[7] M. Mitchell (private communication).