Cold Rydberg Atoms as Realizable Analogs of Chern-Simons Theory

C. Baxter
Department of Physics, University of Essex, Colchester CO4 3SQ, England
(Received 26 July 1994)

The angular momentum spectrum of an atomic dipole in constant electric and magnetic fields is calculated. The arrangement of the fields are such that the motion of the dipole is planar and rotationally symmetric, with the result that the Röntgen term of the Lagrangian takes on the appearance of a Chern-Simons interaction. On eliminating the kinetic energy term of the Lagrangian, the angular momentum spectrum changes from one consisting of integers to one consisting of positive half integers. The possibility of physically achieving such an elimination is discussed in the context of cold Rydberg atoms.

PACS numbers: 42.50.Vk, 11.10.Ef

Physical systems confined to a spacetime of less than four dimensions show a variety of interesting properties. A well-known example is that of the one-dimensional hydrogen atom: its excited bound states are twofold degenerate because of the existence of a pole in the one-dimensional Coulomb potential [1]. Further and perhaps somewhat more exotic instances, such as the quantum Hall effect, high T_c superconductivity, and cosmic strings in planar gravity, have arisen in recent times. In many of these cases, the role of the intrinsically odd-dimensional Chern-Simons interaction is evident. This term was originally constructed out of gauge fields in (1 + 2) dimensions, that is one temporal and two spatial dimensions [2]; the formalism associated with these topologically massive gauge fields was then adapted and used to provide effective field theories for other confined systems [3].

Models of confined systems which are analogous to topologically massive gauge theories have been constructed theoretically; these give useful insights into a class of physics which is unknown in four-dimensional spacetime. For example, consider a charged particle moving in constant external magnetic and electric fields. If the motion is constrained to be planar and rotationally symmetric, then the vector potential component of the minimal coupling takes on the form of a Chern-Simons interaction [4]. A theory analogous to that of "pure Chern-Simons" is obtained in the limit of vanishing particle mass; the removal of the kinetic energy term collapses the Lagrangian to first order and ultimately makes the particle's angular momentum spectrum consists of half integers. The subject of the present Letter is to show that such an analog, but one which is physically realizable, may occur in the case of a cold Rydberg atom. Here, by a suitable experimental arrangement, the Röntgen interaction takes on the appearance of a Chern-Simons term, and the reduction in order of the Lagrangian is achieved in part from the overwhelming importance of the large dipole moment.

A rigorous treatment of the center-of-mass dynamics for a moving dipole in an electromagnetic field reveals the presence of an interaction term linear in the dipole's velocity [5]. The Röntgen interaction, which is consistent with a canonical identification of the center-of-mass Hamiltonian, is necessary in order to conserve momentum and to ensure gauge invariance. Its origin lies in the classical Röntgen current which is generated by the gross motion of any aggregate of charges. The Röntgen interaction makes a small but potentially interesting contribution to laser cooling effects [6]; it yields the correct velocity dependence of atomic optical responses [7], and it is also responsible for inducing a quantum phase in a dipole moving in a constant magnetic field [8].

In three-dimensional space, the Lagrangian

$$L = \frac{M}{2}\dot{\mathbf{R}} - \dot{\mathbf{R}} \cdot \mathbf{d}\times\mathbf{B}(\mathbf{R}) + \mathbf{d} \cdot \mathbf{E}(\mathbf{R}) \qquad (1)$$

represents a dipole of moment $\mathbf{d}$ and mass M, whose position $\mathbf{R}$ in the laboratory frame moves in electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields. The Röntgen energy $\dot{\mathbf{R}}\cdot\mathbf{d} \times \mathbf{B}(\mathbf{R})$ ensures that the canonical momentum $\mathbf{P} = M\dot{\mathbf{R}} - \mathbf{d}\times\mathbf{B}(\mathbf{R})$ differs from its purely mechanical component $M\dot{\mathbf{R}}$. Suppose that the motion of the dipole is restricted to two dimensions $i = 1, 2$ by the application of constant electric and magnetic fields in an appropriate experimental arrangement, such that

$$\dot{\mathbf{R}}\cdot\mathbf{d} \times \mathbf{B}(\mathbf{R}) = \frac{g}{2}\epsilon_{ii'}\dot{R}_i R_{i'}\,, \qquad (2a)$$

$$\mathbf{d} \cdot \mathbf{E}(\mathbf{R}) = -\frac{k}{2}R_i R_i\,. \qquad (2b)$$

In Eq. (2), the usual summation convention over repeated indices has been adopted, and the parameters g and k are constants—the factor of 2 is added for later convenience. The two-dimensional Levi-Civita matrix $\epsilon_{ii'}$ has vanishing diagonal elements, with $\epsilon_{12} = -\epsilon_{21} = 1$. Equation (2) implies that the fields are arranged in a crossed formation with the magnetic field aligned along the z axis and the electric field acting radially in the x-y plane. The Lagrangian of the restricted system is therefore

$$L = \frac{M}{2}\dot{R}_i\dot{R}_i + \frac{g}{2}\epsilon_{ii'}R_i\dot{R}_{i'} - \frac{k}{2}R_iR_i\,. \qquad (3)$$

Confining the dipole to planar, rotationally symmetric motion defined by (2) causes the Röntgen interaction to take on a Chern-Simons appearance $\epsilon_{ii'}R_i\dot{R}_{i'}$. Equation (3) is

0031-9007/95/74(4)/514(4)$06.00

of the same form as the Lagrangian for a charged particle executing planar, rotationally symmetric motion in a constant magnetic field and quadratic scalar potential, and is also analogous to the Lagrangian density for (1 + 2)-dimensional, topologically massive electrodynamics in the Weyl gauge [4].

The orbital angular momentum spectrum of the system described by (3) consists of positive and negative integers, but, by removing the kinetic energy term, the spectrum is made to assume one of only positive half integers. Choosing the dipole to be that of a cold Rydberg atom may provide the physical means of constructing such a first-order Lagrangian. The parameter g is proportional to the magnitude of the dipole moment and dependent on the magnetic field. It therefore assumes a dominant role in the case of a Rydberg atom in a large magnetic field. Further, by constructing the effect of the last term of (3) from an appropriate optical trapping field, the possibility arises of slowing the speed of the atom to the extent that the kinetic energy term may become insignificant [9].

The theory will be described first from the full Lagrangian, where the kinetic energy term is retained and where Lagrange's equation gives $M\ddot{R}_i = g\epsilon_{ii'}\dot{R}_{i'} - kR_i$ as the classical equation of motion. The path to quantization leads straightforwardly from (3) and its conjugate momenta

$$P_i = M\dot{R}_i - \frac{g}{2}\epsilon_{ii'}R_{i'}, \tag{4}$$

together with the fundamental Poisson brackets

$$\{R_i, P_{i'}\} = \delta_{ii'}, \tag{5a}$$

$$\{R_i, R_{i'}\} = \{P_i, P_{i'}\} = 0, \tag{5b}$$

to the Hamiltonian operator

$$\hat{H} = \frac{\hat{P}_i\hat{P}_i}{2M} + \frac{g}{2M}\epsilon_{jj'}\hat{P}_j\hat{R}_{j'} + \frac{M\Omega^2}{2}\hat{R}_k\hat{R}_k \tag{6}$$

and associated commutators

$$[\hat{R}_i, \hat{R}_{i'}] = 0, \tag{7a}$$

$$[\hat{R}_i, \hat{P}_{i'}] = i\hbar\delta_{ii'}. \tag{7b}$$

Quantum mechanical operators have been denoted by carets. The frequency

$$\Omega = \left\{\frac{g^2}{4M^2} + \frac{k}{M}\right\}^{1/2} \tag{8}$$

reveals the dispersive "mass" term $g/2M$, which comes from the presence of the Chern-Simons term in the Lagrangian. In deriving (7), the equivalence

$$[\hat{x}, \hat{y}] \equiv i\hbar\{x, y\} \tag{9}$$

has been used to obtain the commutator between any two quantum mechanical operators $\hat{x}$, $\hat{y}$. By changing the variables to

$$\hat{r}_\pm = \left[\frac{M\Omega}{2\omega_\pm}\right]^{1/2}\hat{R}_1 \mp \left[\frac{1}{2M\Omega\omega_\pm}\right]^{1/2}\hat{P}_2, \tag{10a}$$

$$\hat{p}_\pm = \left[\frac{\omega_\pm}{2M\Omega}\right]^{1/2}\hat{P}_1 \pm \left[\frac{M\Omega\omega_\pm}{2}\right]^{1/2}\hat{R}_2, \tag{10b}$$

where

$$\omega_\pm = \Omega \pm \frac{g}{2M}, \tag{11}$$

the Hamiltonian may be written in the form

$$\hat{H} = \frac{\hat{p}_+^2}{2} + \frac{\hat{p}_-^2}{2} + \frac{\omega_+^2}{2}\hat{r}_+^2 + \frac{\omega_-^2}{2}\hat{r}_-^2 \tag{12}$$

of two uncoupled harmonic oscillators of unit mass and of frequencies $\omega_\pm$. The dimensions of $\hat{p}_\pm$ and $\hat{r}_\pm$ are $[M]^{1/2}[L][T]^{-1}$ and $[M]^{1/2}[L]$, respectively. The Hamiltonian (12) becomes

$$\hat{H} = \hbar\omega_+\left\{\hat{a}_+^\dagger\hat{a}_+ + \tfrac{1}{2}\right\} + \hbar\omega_-\left\{\hat{a}_-^\dagger\hat{a}_- + \tfrac{1}{2}\right\} \tag{13}$$

in the holomorphic representation

$$\hat{r}_\pm = \left[\frac{\hbar}{2\omega_\pm}\right]^{1/2}\{\hat{a}_\pm + \hat{a}_\pm^\dagger\}, \tag{14a}$$

$$\hat{p}_\pm = -i\left[\frac{\hbar\omega_\pm}{2}\right]^{1/2}\{\hat{a}_\pm - \hat{a}_\pm^\dagger\}. \tag{14b}$$

The boson operators $\hat{a}_\pm$, $\hat{a}_\pm^\dagger$ ensure that

$$[\hat{r}_\alpha, \hat{p}_{\alpha'}] = i\hbar\left[\hat{a}_\alpha, \hat{a}_{\alpha'}^\dagger\right] = i\hbar\delta_{\alpha\alpha'}, \tag{15}$$

where $\alpha, \alpha' = +, -$. They annihilate and create $\pm$ number states $|n_+, n'_-\rangle = |n_+\rangle \otimes |n'_-\rangle$ of the dipole, appropriate to the usual quantum harmonic oscillator scheme

$$\hat{a}_+|n_+, n'_-\rangle = \sqrt{n_+}|n_+ - 1, n'_-\rangle, \tag{16a}$$

$$\hat{a}_+^\dagger|n_+, n'_-\rangle = \sqrt{n_+ + 1}|n_+ + 1, n'_-\rangle, \tag{16b}$$

$$\hat{a}_-|n_+, n'_-\rangle = \sqrt{n'_-}|n_+, n'_- - 1\rangle, \tag{16c}$$

$$\hat{a}_-^\dagger|n_+, n'_-\rangle = \sqrt{n'_- + 1}|n_+, n'_- + 1\rangle, \tag{16d}$$

and normalized according to $\langle m'_-, m_+|n_+, n'_-\rangle = \delta_{nm}\delta_{n'm'}$.

Since $g \neq 0$, it is evident from Eq. (11) that $\omega_- \neq \omega_+$; therefore, from the Hamiltonian (13), the energy eigenvalues are nondegenerate providing $\omega_\pm$ is not a multiple of $\omega_\mp$. If, say, $\omega_+ = \zeta\omega_-$, where ζ is an integer greater than 2, then $g = 2M(\zeta - 1)/(\zeta + 1)$. The possibility of a degeneracy occurring by ω_- itself vanishing is prevented by a nonzero k [10].

The advantage of the representation (14) lies in the ease in which it allows the determination to be made of the eigenvalue equation

$$\hat{J}|n_+, n_-\rangle = \hbar(n_- - n_+)|n_+, n_-\rangle \tag{17}$$

for the canonical angular momentum

$$\hat{J} = \epsilon_{ii'}\hat{R}_i\hat{P}_{i'} \tag{18}$$

of the dipole. In (1 + 2) dimensions, angular momentum, like any quantity defined as an exterior product, is a scalar [11]. Equation (17) follows straightforwardly from Eqs. (10), (14)–(16), and

$$\hat{J} = \frac{1}{2}\{\omega_+^{-1}\hat{p}_+^2 + \omega_+\hat{r}_+^2\} - \frac{1}{2}\{\omega_-^{-1}\hat{p}_-^2 + \omega_-\hat{r}_-^2\}$$
$$= \frac{\hbar}{2}\{\hat{a}_-\hat{a}_-^\dagger + \hat{a}_-^\dagger\hat{a}_-\} - \frac{\hbar}{2}\{\hat{a}_+\hat{a}_+^\dagger + \hat{a}_+^\dagger\hat{a}_+\}. \quad (19)$$

Therefore, the angular momentum eigenvalues of the dipole, whose Lagrangian is given by (3), assume the values of any integer multiple of $\hbar$. They are also infinitely degenerate. However, this degeneracy is lifted in the limit of vanishing kinetic energy, where the eigenvalues are only positive half-integer multiples of $\hbar$. This can be seen by expanding (8) by the binomial theorem and using the result with (11). The removal of the kinetic energy term corresponds to taking the limit $M \longrightarrow 0$, where the frequency ω_+ diverges and ω_- tends to k/g. Therefore $\hat{r}_+$ vanishes, to give

$$\lim_{M\longrightarrow 0} \hat{J} = \frac{\hbar}{2}\{\hat{a}_-\hat{a}_-^\dagger + \hat{a}_-^\dagger\hat{a}_-\} \quad (20)$$

and

$$\lim_{M\longrightarrow 0} \hat{J}|n_-\rangle = \hbar\left\{n_- + \frac{1}{2}\right\}|n_-\rangle \quad (21)$$

as the limiting angular momentum and eigenvalue equations. The positive half-integer spectrum comes about because the absence of the + oscillator ensures that the zero-point (vacuum) angular momentum $\hbar/2$ contribution of the − oscillator remains uncanceled.

From the form of the Hamiltonian, it is evident that the canonical angular momentum (18) is a constant of motion. This is not the case for the purely mechanical angular momentum operator

$$\hat{J}_{\text{mech}} = M(i/\hbar)\epsilon_{ii'}\hat{R}_i[\hat{H}, \hat{R}_{i'}]$$
$$= \hat{J} - (g/2)\hat{R}_i\hat{R}_i, \quad (22)$$

except where it vanishes in the limit $M \longrightarrow 0$. The positive half-integer angular momentum spectrum (21) is therefore a consequence of the presence of the Chern-Simons interaction $(g/2)\epsilon_{ii'}R_i\dot{R}_{i'}$, which remains in the Lagrangian after the removal of the kinetic energy term.

Of course, the absence of the kinetic energy term in (3) creates a first-order Lagrangian with primary constraints

$$\phi_i = P_i + \frac{g}{2}\epsilon_{ii'}R_{i'} \approx 0, \quad (23)$$

which are indicative of the momenta not being independent functions. Following the usual procedure [12], the symbol $\approx$ means that all Poisson brackets of interest must be determined before any use is made of the constraints. Remembering that $\epsilon_{ii'} = -\epsilon_{i'i}$, the Poisson bracket

$$\{\phi_i, \phi_{i'}\} = g\epsilon_{ii'} \quad (24)$$

between the constraints is determined. Since this is nonzero, there are therefore no secondary constraints. The equivalence (9) must be replaced by

$$[\hat{x}, \hat{y}] \equiv i\hbar\{x, y\} - i\hbar\{x, \phi_i\}C_{ii'}\{\phi_{i'}, y\}, \quad (25)$$

where the matrix

$$C_{ii'} = -(1/g)\epsilon_{ii'} = (1/g)\epsilon_{i'i} \quad (26)$$

is defined as the inverse of (24). From Eqs. (25), (26), and the Poisson brackets

$$\{R_i, \phi_{i'}\} = \delta_{ii'}, \quad (27a)$$

$$\{P_i, \phi_{i'}\} = (g/2)\epsilon_{ii'}, \quad (27b)$$

together with the identity $\epsilon_{ij}\epsilon_{ji'} = -\delta_{ii'}$, the commutators

$$[\hat{R}_i, \hat{R}_{i'}] = -i(\hbar/g)\epsilon_{ii'}, \quad (28a)$$

$$[\hat{R}_i, \hat{P}_{i'}] = i(\hbar/2)\delta_{ii'} \quad (28b)$$

are found. These commutators differ from those of (7), but are obtained in the limit $M \longrightarrow 0$ from Eqs. (4), (10), and (15) with $\alpha = \alpha' = -$. The departure of the commutators (28) from their usual canonical values is a feature in the quantization of any system governed by a Lagrangian that is linear in velocity. Although this is a standard result [13], it is not well known and for this reason it has been thought necessary to sketch the derivation.

In the absence of the kinetic energy term from the Lagrangian (3), the angular momentum $(g/2)\hat{R}_i\hat{R}_i$ assumes the same form as the Hamiltonian $(k/2)\hat{R}_i\hat{R}_i$, and from (28a) it can be seen to generate rotations in the usual manner: $(i/\hbar)[(g/2)\hat{R}_i\hat{R}_i, \hat{R}_j] = -\epsilon_{ji'}\hat{R}_{i'}$. In the case of the full Lagrangian, such rotations are generated from (18) in conjunction with (7). The equations of motion $d\hat{P}_i/dt = -k\hat{R}_i$, obtained with the aid of (28b), are consistent with the absence of the kinetic energy term from (3). Finally, the representations

$$\hat{R}_1 = (2/g)\hat{P}_2 = \left[\frac{\hbar}{2g}\right]^{1/2}\{\hat{a} + \hat{a}^\dagger\}, \quad (29a)$$

$$\hat{P}_1 = -(g/2)\hat{R}_2 = -\frac{i}{2}\left[\frac{\hbar g}{2}\right]^{1/2}\{\hat{a} - \hat{a}^\dagger\}, \quad (29b)$$

in terms of boson creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators of number states $|n\rangle$, are consistent with (28) and angular momentum eigenvalues of $\hbar(n + [1/2])$.

To induce transitions between the orbital angular momentum eigenlevels, the dipole is now considered to interact with a suitable radiation mode. One could envisage that such a mode might perhaps be of a Laguerre-Gaussian form, since a Laguerre-Gaussian beam carries orbital angular momentum along its direction of propagation [14]. However, for the present purpose of determining transitions dynamics, the radiation mode need not be specified further than its boson annihilation $\hat{b}$ and creation $\hat{b}^\dagger$ operators which form the commutator $[\hat{b}, \hat{b}^\dagger] = 1$. Therefore,

$$\hat{H}' = \sum_{\alpha=+,-}\{\hbar\omega_\alpha\hat{a}_\alpha^\dagger\hat{a}_\alpha + \lambda_\alpha\hat{a}_\alpha\hat{b}^\dagger + \lambda_\alpha^*\hat{a}_\alpha^\dagger\hat{b}\} + \hbar\omega\hat{b}^\dagger\hat{b} \quad (30)$$

will be taken as the total Hamiltonian, in the rotating wave approximation, of the planar-confined dipole coupled to the single-mode radiation of frequency ω. The zero-point energies in (13) do not affect dynamical calculations and are ignored. The interaction strengths λ_α are taken to be complex. Assuming that the planar-confined dipole is prepared in its energy ground state $|0,0\rangle$, the temporal variations of the component operators of (30), determined in the usual manner [15], give

$$\langle \hat{J}(t) \rangle = \frac{2\pi}{\hbar} t[|\lambda_-|^2 \delta(\omega_- - \omega) - |\lambda_+|^2 \delta(\omega_+ - \omega)] \langle \hat{b}^\dagger(0) \hat{b}(0) \rangle \quad (31)$$

as the expectation value of the angular momentum (19) at the limit of large time t. Hence, the analysis suggests that two distinct resonances $+$, $-$ occur at $\omega = \omega_\pm$. As M is reduced, reflecting a diminution in the kinetic energy term, the $-$ resonance occurs at ever greater values of ω, until only the $+$ resonance remains achievable at $\omega = k/g$.

The change in the nature of the orbital angular momentum spectrum of a dipole executing planar, rotationally symmetric motion, from one consisting of integers to one consisting of positive half integers, is a result of the dipole's Lagrangian (3) becoming first order in velocity. This state of affairs comes about through the removal of the kinetic energy term of the Lagrangian, the dipole obtaining in the process a nonvanishing zero-point angular momentum. By choosing the dipole to be that of a cold Rydberg atom in a strong magnetic field, the removal of the kinetic energy term, or at least a noticeable reduction in its influence, could be achieved physically. The degree to which this is successful would be indicated by the location and nature of the possible angular momentum resonances. Much of the role of Chern-Simons theory in predicting interesting physics in the settings of odd-dimensional spacetime, such as those found in planar condensed-matter systems and cosmic strings, is conjecture. The present paper has described a simple $(1 + 2)$-dimensional system which for the first time allows in principle an experimental verification of the Chern-Simons feature of fractional angular momentum. The use of the recently introduced Röntgen energy term allows the Lagrangian of a planar-confined Rydberg atom to exhibit a Chern-Simons coupling.

Since submitting this paper, work has appeared which shows that an unconfined two-level atom moving in a Laguerre-Gaussian beam is subject to a radiation-induced torque, which is proportional to the mode's orbital angular momentum quantum number [16]. This enforces the suggestion made above that a Laguerre-Gaussian beam would supply a suitable probe for the angular momentum resonances indicated by Eq. (31).

[1] R. Loudon, Am. J. Phys. **27**, 649 (1959). Chern-Simons theory is not a feature of this work.

[2] The literature on Chern-Simons theory is large and growing. For general discussions and reviews, see S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982); S. Zhang, T. Hanson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989); R. Jackiw, Nucl. Phys. B (Proc. Suppl.) **18A**, 107 (1990); R. Jackiw and So-Young Pi, Phys. Rev. D **44**, 2524 (1991), and references therein.

[3] For the use of topologically massive gauge fields in providing an effective field theory approach to the quantum Hall effect, see J. Fröhlich and T. Kerler, Nucl. Phys. **B354**, 369 (1991); J. Fröhlich and A. Zee, Institute for Theoretical Physics, Santa Barbara Report No. NSF-ITP-91-31, 1991 (unpublished).

[4] G. V. Dunne, R. Jackiw, and C. A. Trugenberger, Phys. Rev. D **41**, 661 (1990).

[5] C. Baxter, M. Bibiker, and R. Loudon, Phys. Rev. A **47**, 1278 (1993).

[6] V. Lembessis, M. Babiker, C. Baxter, and R. Loudon, Phys. Rev. A **48**, 1594 (1993).

[7] M. Wilkens, Phys. Rev. A **47**, 671 (1993).

[8] M. Wilkens, Phys. Rev. Lett. **72**, 5 (1994).

[9] Atomic velocities of 1 m s^{-1} have been obtained using a number of laser beams and exploiting Zeeman tuning: F. Shimizu, K. Shimizu, and H. Takuma, Opt. Lett. **16**, 339 (1991).

[10] A. P. Polychronakos, Ann. Phys. (N.Y.) **203**, 231 (1990).

[11] S. Forte, Rev. Mod. Phys. **64**, 193 (1992).

[12] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).

[13] L. Faddeev and R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988).

[14] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).

[15] R. Loudon, *The Quantum Theory of Light* (Clarendon, Oxford, 1983), 2nd. ed.

[16] M. Babiker, W. L. Power, and L. Allen, Phys. Rev. Lett. **73**, 1239 (1994).