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Perturbative Renormalization in Quantum Few-Body Problems

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As in quantum field theory, for some singular two-body potentials, the momentum-space perturbative treatment of the three-dimensional quantum N -body problem is shown to exhibit ultraviolet divergences. Renormalization of these models leads to a scale and finite observables. For a Dirac delta potential we carry on this renormalization for two- and three-body systems. A new divergence, and hence a new scale, emerges for each added particle. For $N \geq 3$ there are no divergences in one and two dimensions. The physical implications of these results are discussed.

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The ultraviolet divergences in perturbative quantum field theory can be eliminated by renormalization to define physical observables, such as charge or mass [1,2]. Ultraviolet divergences also appear in the perturbative treatment of the nonrelativistic quantum mechanical two-body problem interacting via two-body potentials with certain singular behavior at short distances [3–7] in two and three space dimensions. Renormalization of these potential models leads to a scale and finite physical observables. In one space dimension these divergences are absent.

In this Letter we show that ultraviolet divergences also appear in the perturbative treatment of the nonrelativistic quantum mechanical N -body ($N > 2$) problem in three space dimensions interacting via two-body potentials with certain singular behavior at short distances. Renormalization of these models again leads to new scales as the number of particles are increased. To the best of our knowledge this is the first investigation of renormalization of the quantum N -body ($N > 2$) problem. In one and two space dimensions there are no such divergences.

Recently, there have been discussions on perturbative renormalization in configuration [5,6] and momentum [3,7] spaces for the quantum two-body problem with a Dirac delta, contact, or zero-range potential. Here we work in momentum space with this pair potential, so that the

treatment remains under control for $N > 2$. This potential is simple, local, and separable at the same time, and has been used in atomic, particle [7,8], nuclear [9], and surface physics [5,6] in both configuration and momentum spaces. This potential has also frequently been used in one space dimension [10]. Though the present renormalization scheme can be carried out in the case of other potentials singular at short distances, the Dirac delta potential has the advantage of permitting a complete analytical treatment for the two-body system.

In three space dimensions the perturbative treatment of the three-body problem exhibits new ultraviolet divergences. This problem can be renormalized by the introduction of a new scale, which is the binding energy of the three-body ground state. For $N > 2$ and space dimension $\mathcal{D} = 3$, a new divergence, and hence a new scale, appears for each added particle. However, once the two-body problem is renormalized, no new divergence appears for $N > 2$ and $\mathcal{D} < 3$ and there is no need for renormalization.

We first discuss the two-body problem for the S -wave delta potential. The partial-wave Lippmann-Schwinger equation for the scattering amplitude $t(p, q, k^2)$ in \mathcal{D} space dimensions at two-body c.m. energy k^2 is given by

$$t(p', p, k^2) = V(p', p) + \int d^{\mathcal{D}}q V(p', q) \times g(q; k^2)t(q, p, k^2), \quad (1)$$

with the free Green function

$$g(q; k^2) = (k^2 - q^2 + i0)^{-1}, \quad (2)$$

in units $\hbar = m = 1$, where m is the mass of each of the particles. The integral in Eq. (1) and in the following is over the full phase space. Throughout this Letter the energy is supposed to contain a small positive imaginary part $i0$, as in Eq. (2), which is suppressed in the following. For the delta potential $V(p', p) = \lambda$, and

$$t(p', p, k^2) = [\lambda^{-1} - I(k^2)]^{-1}, \quad (3)$$

with

$$I(k^2) = \int d^{\mathcal{D}}q g(q; k^2). \quad (4)$$

It is noted that the following binomial expansion of Eq. (3) is the Born series in this case

$$t(p', p, k^2) = \lambda + \lambda^2 I(k^2) + \lambda^3 [I(k^2)]^2 + \dots, \quad (5)$$

and that $\lambda I(k^2)$ is the trace of the kernel of the integral equation (1) and possesses ultraviolet divergence for $\mathcal{D} > 1$. The kernel of Eq. (1) is noncompact and it does not have a scattering solution. The perturbative series (5) is divergent order by order for $\mathcal{D} > 1$ and some regularization is needed to give meaning to Eq. (1). This could be achieved via the following regularized free Green function:

$$g_R(q; k^2) = (k^2 - q^2)^{-1} + (\Lambda_2^2 + q^2)^{-1} \\ = \frac{k^2 + \Lambda_2^2}{(k^2 - q^2)(\Lambda_2^2 + q^2)}. \quad (6)$$

The imaginary part of the Green function is unaffected by this procedure which guarantees unitarity. With this regularized Green function one has for the regularized t matrix

$$t_R(p', p, k^2) = [\lambda_R^{-1} - I_R(k^2)]^{-1}, \quad (7)$$

where

$$I_R(k^2) = \int d^{\mathcal{D}}q g_R(q; k^2) \quad (8)$$

is a convergent integral. In order that Eq. (3) yields a finite nonzero t matrix, λ^{-1} should diverge, so that the finite nonzero renormalized coupling λ_R is given by

$$\lambda_R^{-1} = \lambda^{-1} - [I(k^2) - I_R(k^2)].$$

Note that the term involving Λ_2 has been introduced in the regularized Green function to eliminate the ultraviolet divergence.

However, now there is a new parameter Λ_2 in the theory, in addition to the coupling λ_R . Both these parameters can

be eliminated in the two-body renormalized t matrix, in favor of a physical observable, e.g., the binding energy of the two-body ground state. This procedure is carried out analytically in the present model. In more complex situations, such as in the three-body problem, one has to rely on numerical means. The condition for a bound state at energy $-\alpha^2$ is given by $\lambda_R^{-1} = I_R(-\alpha^2)$, so that one can define the following renormalized t matrix after eliminating the unknown parameters in terms of the binding energy:

$$t_R(p', p, k^2) = [I_R(-\alpha^2) - I_R(k^2)]^{-1} \\ = - \left[\int \frac{d^{\mathcal{D}}p (k^2 + \alpha^2)}{(p^2 + \alpha^2)(k^2 - p^2)} \right]^{-1} \\ = [2\pi^2(ik + \alpha)]^{-1}, \quad \mathcal{D} = 3. \quad (9)$$

The binding energy α^2 is the energy scale of the two-body system. Once α^2 is given the complete solution of the problem is known.

The final renormalized result (9) should be independent of the detailed renormalization scheme. One could have used the following regularized Green function in place of the one given by Eq. (6):

$$g_R(q; k^2) = (k^2 - q^2)^{-1} \Theta(\Lambda_2 - q), \quad (10)$$

with $\Lambda_2 > k$, where $\Theta(x) = 0$ for $x < 0$ and $=1$ for $x > 0$. With this regularized Green function the regularized t matrix is again given by Eqs. (7) and (8) but now with $g_R(q; k^2)$ given by Eq. (10). The remarkable result is that, even with this regularization procedure, the constants λ_R and Λ_2 could be eliminated in favor of the two-body binding energy and one recovers the renormalized t matrix given by Eq. (9).

Next let us consider the three-body problem with the above two-body delta potential. One encounters new divergences in this case. The perturbative series for the full three-to-three amplitude $T(E)$ is given by

$$T(E) \equiv \sum_{i=1}^3 V_{(i)} + \sum_{i=1}^3 V_{(i)} G_0(E) \sum_{j=1}^3 V_{(j)} \\ + \sum_{i=1}^3 V_{(i)} G_0(E) \sum_{j=1}^3 V_{(j)} G_0(E) \sum_{l=1}^3 V_{(l)} + \dots \quad (11)$$

The potential $V_{(i)} = \lambda$ ($i = 1, 2, 3$) in this series is for the "pair i " of particles j and l ($i \neq j \neq l \neq i$), $G_0(E) = (E - H_0)^{-1}$ is the three-body free Green function at three-body c.m. energy E with H_0 the free Hamiltonian. This series develops new ultraviolet divergences in all terms starting from the fourth order term for $\mathcal{D} = 3$. For $\mathcal{D} < 3$ all terms in this series are finite. A typical fourth order term in series (11) is given by

$$\iint \frac{\lambda^4 d^{\mathcal{D}}q d^{\mathcal{D}}q'}{(E - p^2 - q^2 - \vec{p} \cdot \vec{q})(E - q'^2 - q^2 - \vec{q}' \cdot \vec{q})(E - p'^2 - q'^2 - \vec{p}' \cdot \vec{q}')} \quad (12)$$

where \vec{p} and \vec{p}' are the momenta of the initial and final spectator particles, respectively. For $\mathcal{D} = 1$ or 2, the integral (12) is convergent at the upper limit. For $\mathcal{D} = 3$ this integral diverges logarithmically at the upper limit. As we increase the order of terms in series (11) we shall have one more loop integral and one Green function. For $\mathcal{D} = 1$, for each added loop integral, we have one extra power of momentum in the numerator and for the Green function two extra powers of momenta in the denominator, consequently, this will make the higher order terms of the multiple scattering series more convergent. For $\mathcal{D} = 2$, for each added loop integral, we shall have two extra powers of momenta in the numerator and for the Green function two extra powers of momenta in the denominator and this will not alter the convergence rate of the higher order terms at the upper limit. However, for $\mathcal{D} = 3$, for each added loop integral we shall have three extra powers of momenta in the numerator and for the additional Green function two extra powers of momenta in the denominator and this will make the higher order terms more and more divergent at the upper limit.

The ultraviolet divergences for $\mathcal{D} = 3$ can again be removed by using the following regularized three-body Green function $G_R(E)$:

$$G_R(E) \equiv G_0(E) + \hat{G}_0(E) = \frac{E + \Lambda_3^2}{(E - H_0)(\Lambda_3^2 + H_0)}, \quad (13)$$

where $\hat{G}_0(E) = (\Lambda_3^2 + H_0)^{-1}$ and Λ_3 is a new arbitrary constant. With this Green function there will be the needed extra powers of momenta in the denominator of Eq. (12) to ensure convergence.

Now remains the nontrivial task of summing up the perturbative series (11) and constructing the renormalized solution to the three-body problem. For this we rely on the Faddeev prescription. In this case the three-body connected kernel Faddeev equations are written in terms of the renormalized two-body t matrix and the free Green function as [11,12]

$$T_i(E) = t_{\mathcal{R}(i)} + t_{\mathcal{R}(i)}G_0(E) \sum_{j \neq i} T_j(E), \quad (14)$$

where $t_{\mathcal{R}(i)}$ is the renormalized two-body t matrix for the pair i in the three-body space. Here $T_i(E)$ contains that part of the three-to-three amplitude which begins with the two-body t matrix $t_{\mathcal{R}(i)}$ and $T(E) \equiv \sum_i T_i(E)$. In order that this equation has a unique solution it should have a compact kernel [11–14]. Then the Fredholm alternative is valid and Eq. (14) can be solved numerically [13,14]. The formal Fredholm solution is constructed in terms of traces of different powers of the kernel [14] and, if these traces diverge, K is noncompact and Eq. (14) does not have a unique solution. For three identical bosons, each of unit mass, the symmetrized kernel of this equation for $\mathcal{D} = 3$ is [11,12]

$$K(\vec{p}, \vec{q}) \sim f(p)(E - p^2 - q^2 - \vec{p} \cdot \vec{q})^{-1}f(q),$$

where $f(p) = [\alpha + (3p^2/4 - E)^{1/2}]^{-1/2}$. The quantity $f^2(p)$ is essentially the two-body t matrix of Eq. (9) in the three-body space with the spectator particle carrying a momentum p . The trace of this kernel, given by

$$\text{Tr}K \sim \int d^3q (E - 3q^2)^{-1}f^2(q),$$

possesses logarithmic ultraviolet divergence which cannot be removed by iteration. For each iteration one introduces three extra powers of momenta in the numerator and three powers of momenta in the denominator for large momenta. So $\text{Tr}K^n$ diverges for all n , and K is not compact. Thus for $\mathcal{D} = 3$ the Faddeev equations (14) do not have a unique solution. For $\mathcal{D} = 1$ and 2 there is no ultraviolet divergence and the kernel K is compact.

In the three-body problem the renormalization of the two-body t matrix is not enough to yield Faddeev equations with compact kernel. In order to achieve compactness we use the regularized Green function (13) to produce the regularized Faddeev equations

$$T_{\mathcal{R}(i)}(E) = t_{\mathcal{R}(i)} + t_{\mathcal{R}(i)}G_R(E) \sum_{j \neq i} T_{\mathcal{R}(j)}(E). \quad (15)$$

For three identical bosons in three dimensions the symmetrized and regularized kernel is given by

$$K_R(\vec{p}, \vec{q}) \sim \frac{f(p)(E + \Lambda_3^2)f(q)}{(\Lambda_3^2 + p^2 + q^2 + \vec{p} \cdot \vec{q})(E - p^2 - q^2 - \vec{p} \cdot \vec{q})}.$$

In order that Eq. (15) has a unique solution the kernel K_R has to be compact. Weinberg has demonstrated that such a kernel is compact if (but not only if) it is an L^2 or Hilbert-Schmidt kernel, i.e., $\text{Tr}(KK^\dagger) < \infty$ [13]. Compared to the kernel $K(\vec{p}, \vec{q})$, $K_R(\vec{p}, \vec{q})$ has two extra powers of momenta in the denominator in the ultraviolet limit, and this makes $\text{Tr}(KK^\dagger)$ finite which is a sufficient condition for compactness for the class of kernels of two- and three-body scattering equations as discussed by Weinberg.

The regularized Faddeev equation (15) corresponds to a fully convergent dynamical theory with the regularized

Green function. The renormalized three-body t matrices can be obtained from its solution by eliminating, as in the two-body problem, the parameter Λ_3 in terms of a three-body observable, e.g., the binding energy of the three-body ground state. However, unlike in the two-body case, this can only be achieved numerically. The same result could have been obtained by using a different regularization procedure, such as by introducing a cutoff in the Green function as in the two-body problem. After elimination of the arbitrary parameter in the regularized Green function in favor of the three-body ground-state

binding energy, the result for the renormalized three-body t matrix should be independent of the regularization scheme as in the two-body problem.

Now it is not difficult to show that in three space dimensions ultraviolet divergences continue appearing in the N -body problem as the number of particles increases beyond three. Regularization of the Green function removes these ultraviolet divergences. However, to construct the renormalized scattering amplitude one has to use the N -body ($N > 3$) connected kernel approach [11], and this is beyond the scope of the present study. Renormalization of this problem leads to a new scale to be fixed by a N -body observable, for example, the binding energy. Introduction of each new particle will lead to new divergences in three space dimensions needing for a new scale to emerge.

In the one-dimensional N -body problem with pair delta potentials there are no ultraviolet divergences and the integrals of the perturbative series are convergent. Hence the binding energy of the N -body system should be determined by the binding energy of the two-body system independent of the details of the potential. For the delta potential, supporting a two-body bound state of binding ϵ , the three-body binding energy is 4ϵ [10].

For $\mathcal{D} = 2$ there are ultraviolet divergences in the perturbative series for $N = 2$ with pair delta potentials. However, for the N -body ($N > 2$) problem there are no ultraviolet divergences. Hence no new scale emerges in the few-body problem for $\mathcal{D} = 2$ and the binding energy of the N -body ($N > 2$) ground state is determined by the binding energy of the two-body ground state. But as the integrals in this case are weakly convergent compared to the one-dimensional case, in calculations employing pair short-range potentials, the sensitivity to the potential will be stronger than in one dimension and the N -body ground-state binding energy will show some dependence on potential models [15].

For $\mathcal{D} = 3$ we encounter ultraviolet divergences for any N for pair delta potentials. After a renormalization of the two-body problem, the three-body Faddeev equation (14) does not have a compact kernel and further regularization of the three-body Green function is needed to achieve compactness via Eq. (15). Then for each added particle new ultraviolet divergences appear, which calls for renormalization. A new energy scale appears at each stage. This makes the three-body ground-state binding energy independent of the two-body ground-state binding energy, the four-body ground-state binding energy independent of two- and three-body ground-state binding energies, and so on. This has interesting consequences.

The three-body ground-state binding energy is the new scale of the three-body problem. This was observed by Thomas [9] in the study of the three-body problem employing a delta potential in three space dimensions using the framework of the time-independent Schrödinger equation. He found that the three-body ground-state

binding energy for a delta potential does not depend on the two-body binding and can become infinite even when the two-body ground-state binding energy is a small finite quantity. This so-called Thomas effect is absent in one and two dimensions, where the three-body ground-state binding energy is determined by the two-body ground-state binding energy [10,15].

However, once the three-body ground-state binding energy or the only new scale of the three-body problem is given, other three-body observables will be uniquely determined and correlated with the three-body binding energy. Many such correlations have been observed in realistic calculations of the three-nucleon system [11,16]. Various nucleon-nucleon short-range potential models with the same two-nucleon binding have the same two-body scale but usually lead to different three-body scales and hence to different three-nucleon binding energies.

In summary, we have exhibited the existence of ultraviolet divergences in the quantum N -body problem, interacting via two-body potentials with certain singularity at short distances, in three space dimensions. Renormalization of these potential models leads to one new scale for each added particle. We explicitly renormalize the two- and three-body problems with pair delta potentials. Such divergences are absent in one and two space dimensions in the N -body ($N > 2$) problem. In three space dimensions $N - 1$ new scales emerge as a result of renormalization of the N -body problem. These $N - 1$ new scales are $N - 1$ physical observables, for example, the i -body binding energies ($i = 2, \dots, N$), and once these energies are given other N -body observables will be determined uniquely.

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